Handling complicated Wick contractions in lattice QCD

– K→ππ part 3 –

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BNL HET Lunch Seminar
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K → ππ & CP violation

CP odd CP even

$|K_L\rangle = |K_2\rangle + \varepsilon |K_1\rangle$

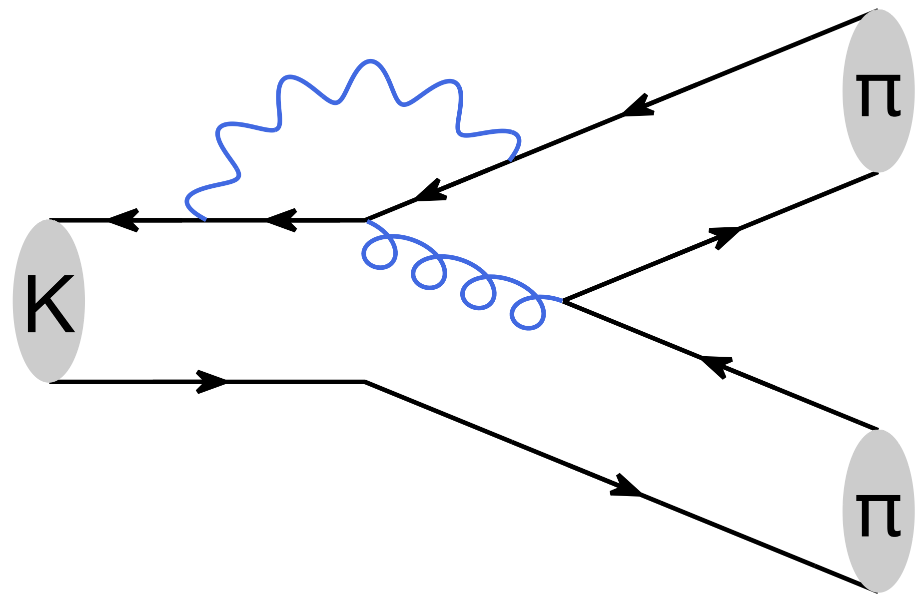
ε' direct CPV

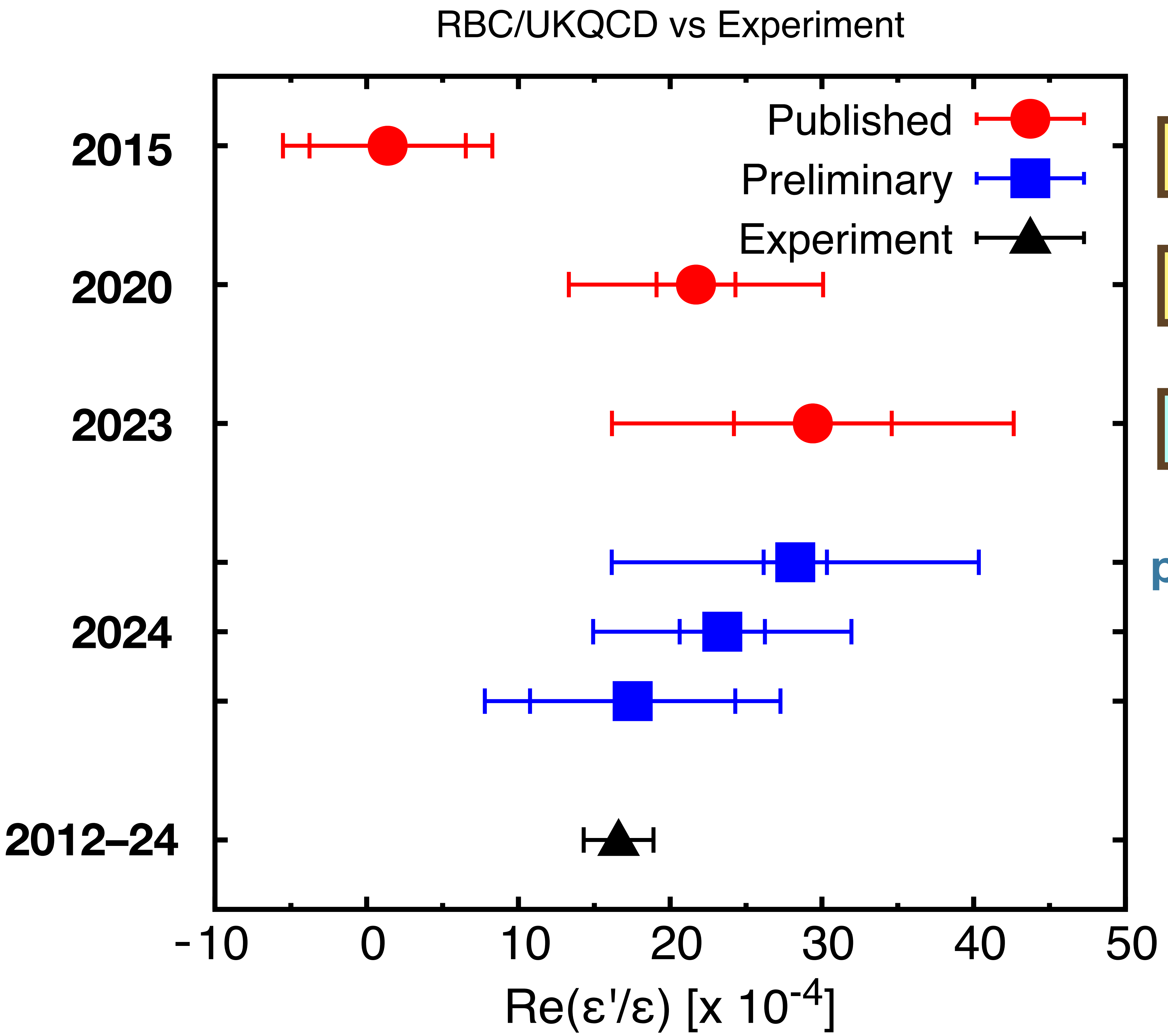
ε indirect CPV

$|\pi\pi\rangle$

CP even

$$\frac{\Gamma(K_L \to \pi^0\pi^0)}{\Gamma(K_S \to \pi^0\pi^0)} \Big/ \frac{\Gamma(K_L \to \pi^+\pi^-)}{\Gamma(K_S \to \pi^+\pi^-)} = 1 - 6\mathrm{Re}(\varepsilon'/\varepsilon)$$





PRL 115,212001

PRD 102,054509

PRD 108,094517

preliminary results with multiple lattice spacings

Challenges

- Operator mixing
  - 10 four-quark operators
  - desired to prevent extra mixing
  - chiral symmetry with domain wall fermions preferable

- On-shell kinematics in euclidean space
  - final two-pion state with $E = m_K \approx 500$ MeV needed

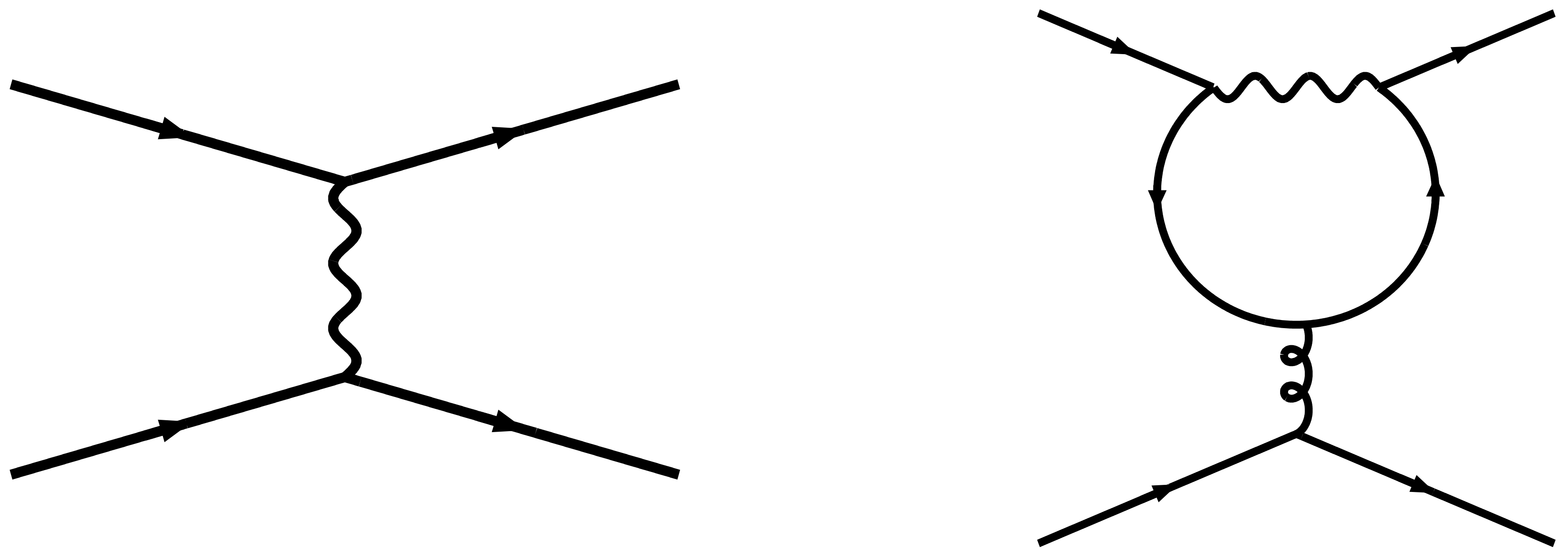
- Computational cost/ Statistics
  - disconnected diagrams

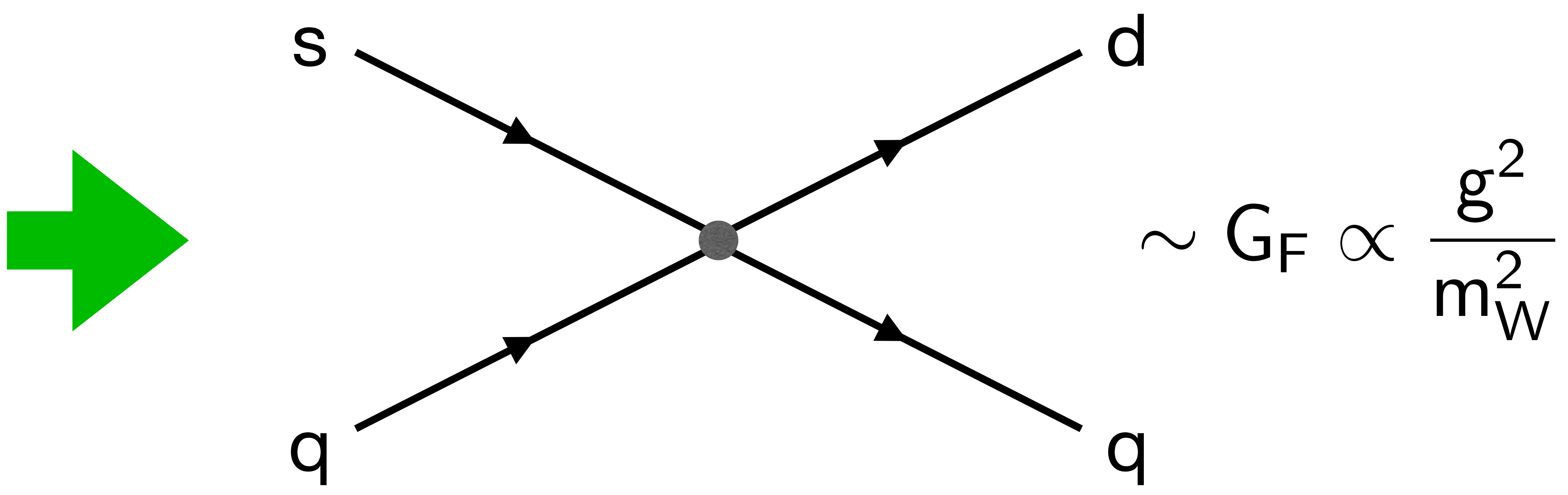
- Charm-loop effects
  - expected significant
  - lattices still not fine enough to introduce charm with physical m_π

- Overall complexity
  - requires significant human time & effort

Approach to weak matrix elements

- Two typical scales
  - Electroweak scale: m_W = 80 GeV, m_Z = 91 GeV
  - QCD scale: Λ_{QCD} ~ 300 MeV
- Low-energy effective interactions @ QCD scale





  - $H_W = \Sigma_i\, c_i(\mu)\, O_i(\mu)$

Wilson coefficients Effective operators

ΔS = 1 effective operators

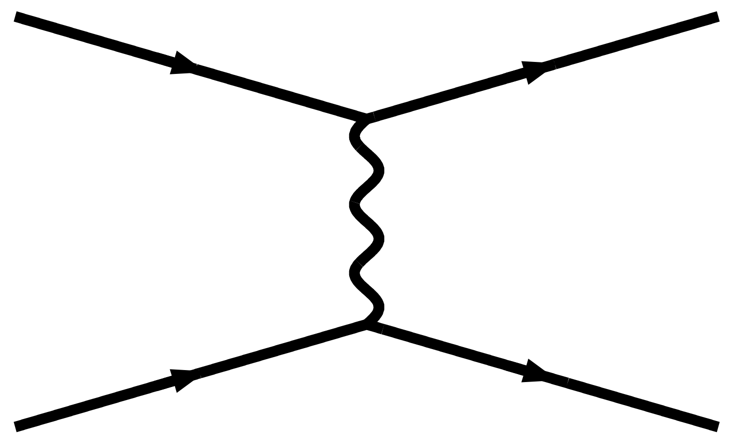
- $(\bar{s}q)_{V-A}(\bar{q}'q'')_{V\pm A} = \bar{s}\gamma_\mu(1-\gamma_5)q' \cdot \bar{q}'\gamma_\mu(1\pm\gamma_5)q''$
- α, β: color indices

$$Q_1 = (\bar{s}_\alpha u_\beta)_{\mathrm{V-A}}(\bar{u}_\beta d_\alpha)_{\mathrm{V-A}},$$
$$Q_2 = (\bar{s}u)_{\mathrm{V-A}}(\bar{u}d)_{\mathrm{V-A}},$$

Current-current operators

- $Q_1^c = (\bar{s}_\alpha c_\beta)_{\mathrm{V-A}}(\bar{c}_\beta d_\alpha)_{\mathrm{V-A}}$ & $Q_2^c = (\bar{s}c)_{\mathrm{V-A}}(\bar{c}d)_{\mathrm{V-A}}$

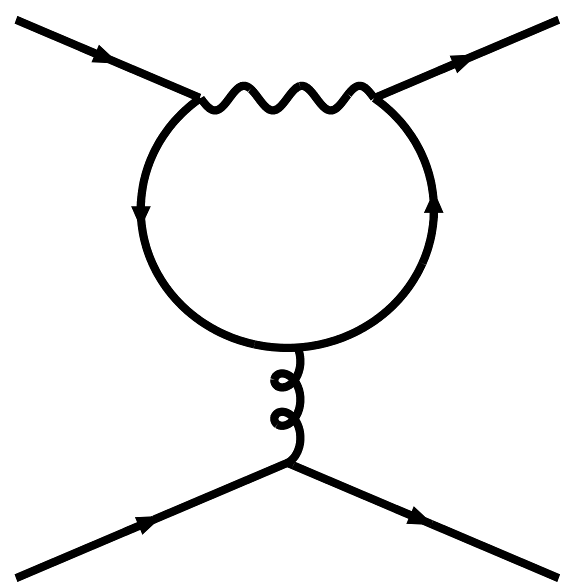
  enter when $n_f \geq 4$



$$Q_3 = (\bar{s}d)_{\mathrm{V-A}}\sum_q(\bar{q}q)_{\mathrm{V-A}},$$
$$Q_4 = (\bar{s}_\alpha d_\beta)_{\mathrm{V-A}}\sum_q(\bar{q}_\beta q_\alpha)_{\mathrm{V-A}},$$
$$Q_5 = (\bar{s}d)_{\mathrm{V-A}}\sum_q(\bar{q}q)_{\mathrm{V+A}},$$
$$Q_6 = (\bar{s}_\alpha d_\beta)_{\mathrm{V-A}}\sum_q(\bar{q}_\beta q_\alpha)_{\mathrm{V+A}},$$

QCD penguin operators

- sum over q runs for all active quarks



$$Q_7 = \frac{3}{2}(\bar{s}d)_{\mathrm{V-A}}\sum_q e_q(\bar{q}q)_{\mathrm{V+A}},$$
$$Q_8 = \frac{3}{2}(\bar{s}_\alpha d_\beta)_{\mathrm{V-A}}\sum_q e_q(\bar{q}_\beta q_\alpha)_{\mathrm{V+A}},$$
$$Q_9 = \frac{3}{2}(\bar{s}d)_{\mathrm{V-A}}\sum_q e_q(\bar{q}q)_{\mathrm{V-A}},$$
$$Q_{10} = \frac{3}{2}(\bar{s}_\alpha d_\beta)_{\mathrm{V-A}}\sum_q e_q(\bar{q}_\beta q_\alpha)_{\mathrm{V-A}},$$

EW penguin operators

$K \to \pi\pi$ Amplitude and ε'

ππ phase shifts at m_K

$$\varepsilon' = \frac{i\omega e^{i(\delta_2-\delta_0)}}{\sqrt{2}}\left[\frac{\mathrm{Im}A_2}{\mathrm{Re}A_2} - \frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}\right] \qquad (\omega = \mathrm{Re}A_2/\mathrm{Re}A_0)$$

$$A_I = \frac{G}{\sqrt{2}} V_{us}^* V_{ud} \sum_i \underbrace{[z_i(\mu) + \tau y_i(\mu)]}_{} M_{I,i}(\mu)$$

Wilson coefs.
pQCD

$$M_{I,i}(\mu) = \lim_{a\to 0} \sum_j Z_{ij}(\mu, a) \left\langle (\pi\pi)_I \middle| Q_j^{lat}(a) \middle| K \right\rangle$$

Bare Matrix elements
LQCD

$K \to \pi\pi$ Amplitude and ε'

ππ phase shifts at m_K

$$\varepsilon' = \frac{i\omega e^{i(\delta_2-\delta_0)}}{\sqrt{2}}\left[\frac{\mathrm{Im}A_2}{\mathrm{Re}A_2} - \frac{\mathrm{Im}A_0}{\mathrm{Re}A_0}\right] \qquad (\omega = \mathrm{Re}A_2/\mathrm{Re}A_0)$$

$$A_I = \frac{G}{\sqrt{2}} V_{us}^* V_{ud} \sum_i [z_i(\mu) + \tau y_i(\mu)] M_{I,i}(\mu)$$

Wilson coefs.
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$$M_{I,i}(\mu) = \lim_{a\to 0} \sum_j Z_{ij}(\mu, a) \left\langle (\pi\pi)_I \middle| Q_j^{lat}(a) \middle| K \right\rangle$$

Renormalization
LQCD+pQCD
(my previous talk)

Bare Matrix elements
LQCD

Outline

- Role of correlation functions (for calculation of bare matrix elements)
  - Basics and some complications for the $K \to \pi\pi$ case
- Basics of how to calculate correlation functions
- Handling complicated calculations – Automated Wick contractor
- Detailed strategy for $K \to \pi\pi$ calculation

Role of correlation functions

for calculation of bare matrix elements

3pt function

want ME of this

$$\langle \underbrace{O_{\mathrm{sink}}}_{\text{annihilates the final state}}(t_1+t_2)\,\underline{Q}(t_1)\,\underbrace{O_{\mathrm{src}}}_{\text{generates the initial state}}(0)^\dagger \rangle$$

source & sink operators usually projected to

- specific quantum numbers, e.g. isospin, parity, strangeness, …
- specific (total) spatial momentum

→ exclude extra states that are out of interest

3pt function

$$\left\langle O^{\varphi}_{\mathrm{sink}}(t_1+t_2)Q(t_1)O^{\varphi'}_{\mathrm{src}}(0)^{\dagger}\right\rangle$$

$$=\sum_{m,n}\left\langle 0\middle|O^{\varphi}_{\mathrm{sink}}\middle|E^{\varphi}_{m}\right\rangle\left\langle E^{\varphi}_{m}\middle|Q\middle|E^{\varphi'}_{n}\right\rangle\left\langle E^{\varphi'}_{n}\middle|O^{\varphi'\,\dagger}_{\mathrm{src}}\middle|0\right\rangle e^{-E^{\varphi}_{m}t_2}e^{-E^{\varphi'}_{n}t_1}$$

ME w ground initial & final states

large t_1 & t_2 →

$$\left\langle 0\middle|O^{\varphi}_{\mathrm{sink}}\middle|E^{\varphi}_{0}\right\rangle\left\langle E^{\varphi}_{0}\middle|Q\middle|E^{\varphi'}_{0}\right\rangle\left\langle E^{\varphi'}_{0}\middle|O^{\varphi'\,\dagger}_{\mathrm{src}}\middle|0\right\rangle e^{-E^{\varphi}_{0}t_2}e^{-E^{\varphi'}_{0}t_1}$$

$\left\langle 0\middle|O^{\varphi}_{\mathrm{sink}}\middle|E^{\varphi}_{0}\right\rangle$ & E^{φ}_{0} obtained from 2pt function

$$\left\langle O^{\varphi}_{\mathrm{sink}}(t)O^{\varphi}_{\mathrm{sink}}(0)^{\dagger}\right\rangle \xrightarrow{\text{large } t} \left\langle 0\middle|O^{\varphi}_{\mathrm{sink}}\middle|E^{\varphi}_{0}\right\rangle\left\langle E^{\varphi}_{0}\middle|O^{\varphi\,\dagger}_{\mathrm{sink}}\middle|0\right\rangle e^{-E^{\varphi}_{0}t}$$

K → ππ case

$$\left\langle O^{\pi\pi}(t_1+t_2)Q_i(t_1)O_K(0)^\dagger\right\rangle$$

$$\to\left\langle 0\middle|O^{\pi\pi}\middle|E_0^{\pi\pi}\right\rangle\underline{\left\langle E_0^{\pi\pi}(\approx 2m_\pi)\middle|Q_i\middle|m_K\right\rangle}\left\langle m_K\middle|O_K^\dagger\middle|0\right\rangle e^{-E_0^{\pi\pi}t_2}e^{-m_K t_1}$$

energy not conserved!

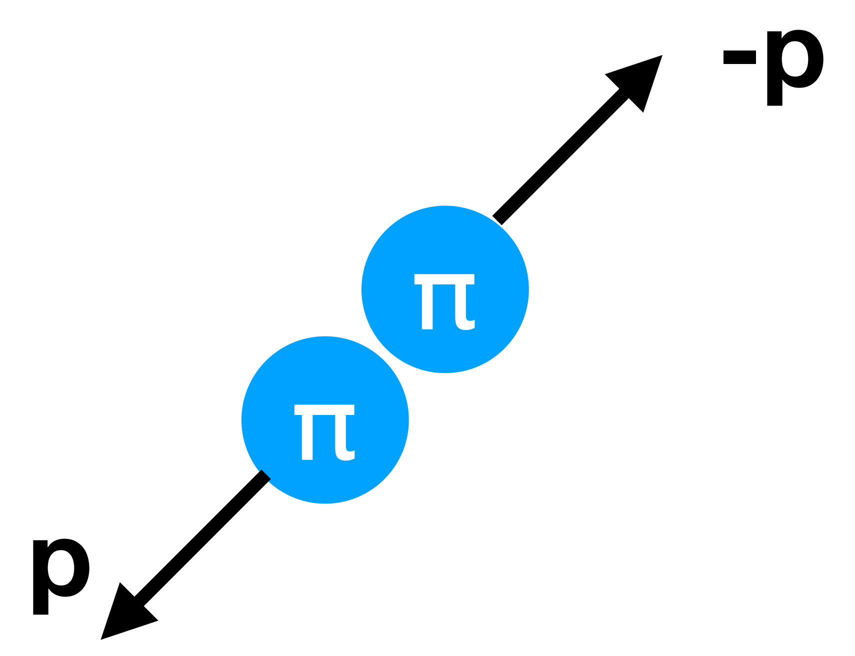
- Need a ππ state with back-to-back momentum
  - G-parity boundary conditions → pions anti-periodic and must move in a finite box
  - Periodic boundary conditions with variational method (my earlier talk)
  - Moving frame also possible in a finite box (1 pion same mom as kaon)

Various ππ operators

- ππ-like operators

$$O^{\pi\pi}(\vec{p}) = \sum_{\vec{x},\vec{y}} e^{i\vec{p}\cdot(\vec{x}-\vec{y})} O_\pi(\vec{x},t) O_\pi(\vec{y},t)$$

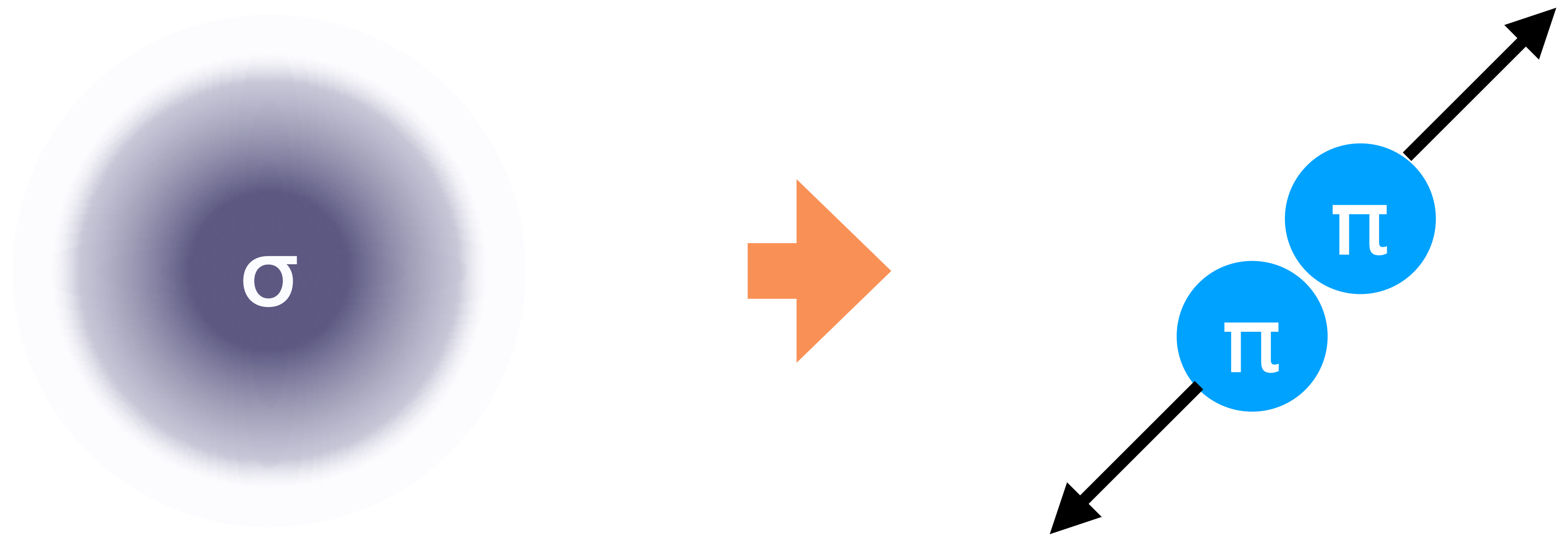
  - $\vec{p} = \vec{0}$: couples well with ground state
  - $\vec{p} = (0,0,\frac{2\pi}{L}),\ (0,\frac{2\pi}{L},\frac{2\pi}{L}),\ (\frac{2\pi}{L},\frac{2\pi}{L},\frac{2\pi}{L})\ \ldots$ control excited states



When isospin is 0:

- σ-like operator to control ππ states near $f_0(500)$ resonance

$$O_\sigma \sim \frac{\bar{u}u + \bar{d}d}{\sqrt{2}}$$

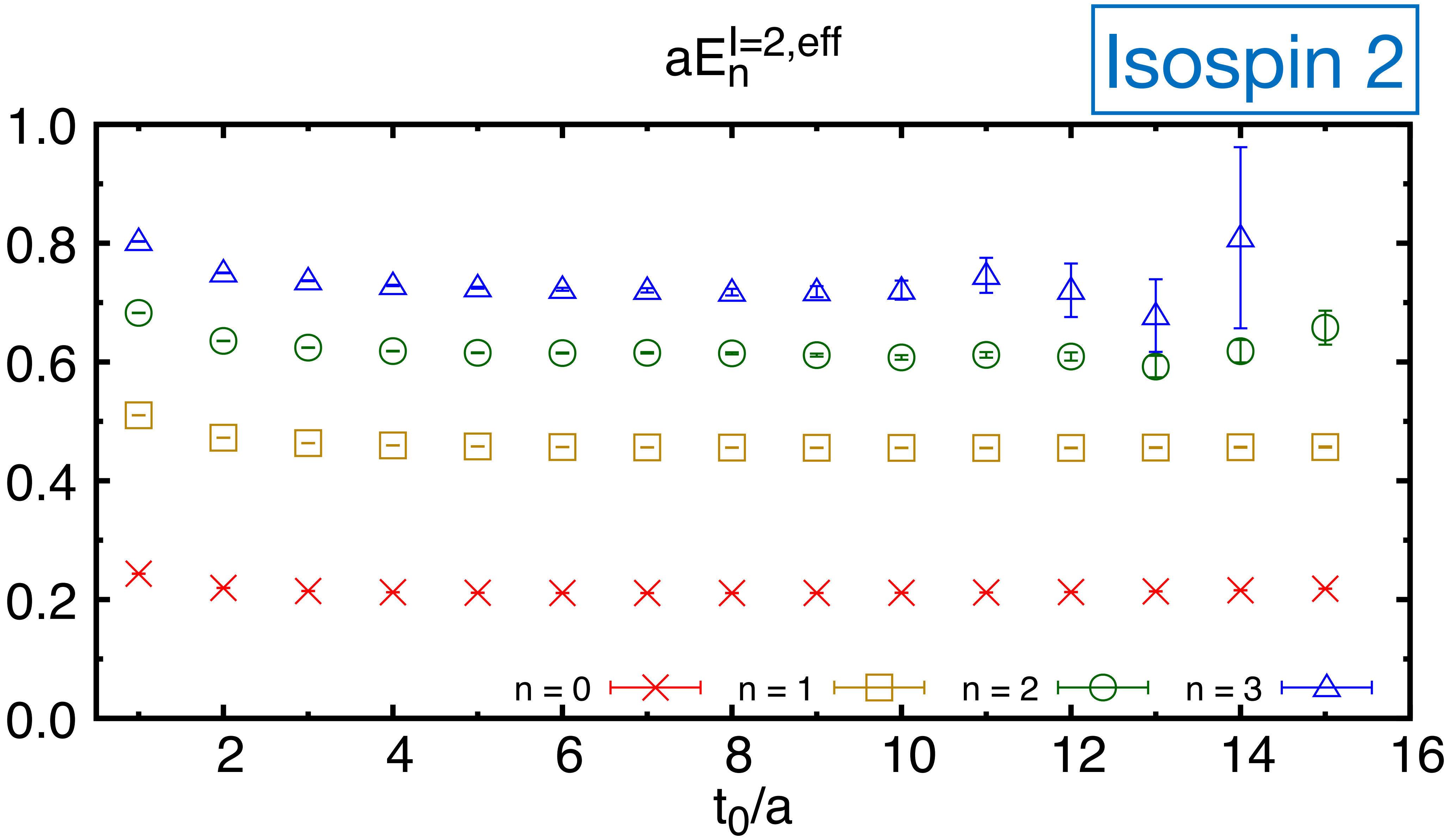


Variational method

$$C_{ab}(t) = \left\langle O_a(t) O_b(0)^\dagger \right\rangle$$: matrix of correlation functions of ππ operators

GEVP ⟶ $$O_n = \sum_a v_{n,a} O_a$$: couple mostly with the E_n state, not with other states

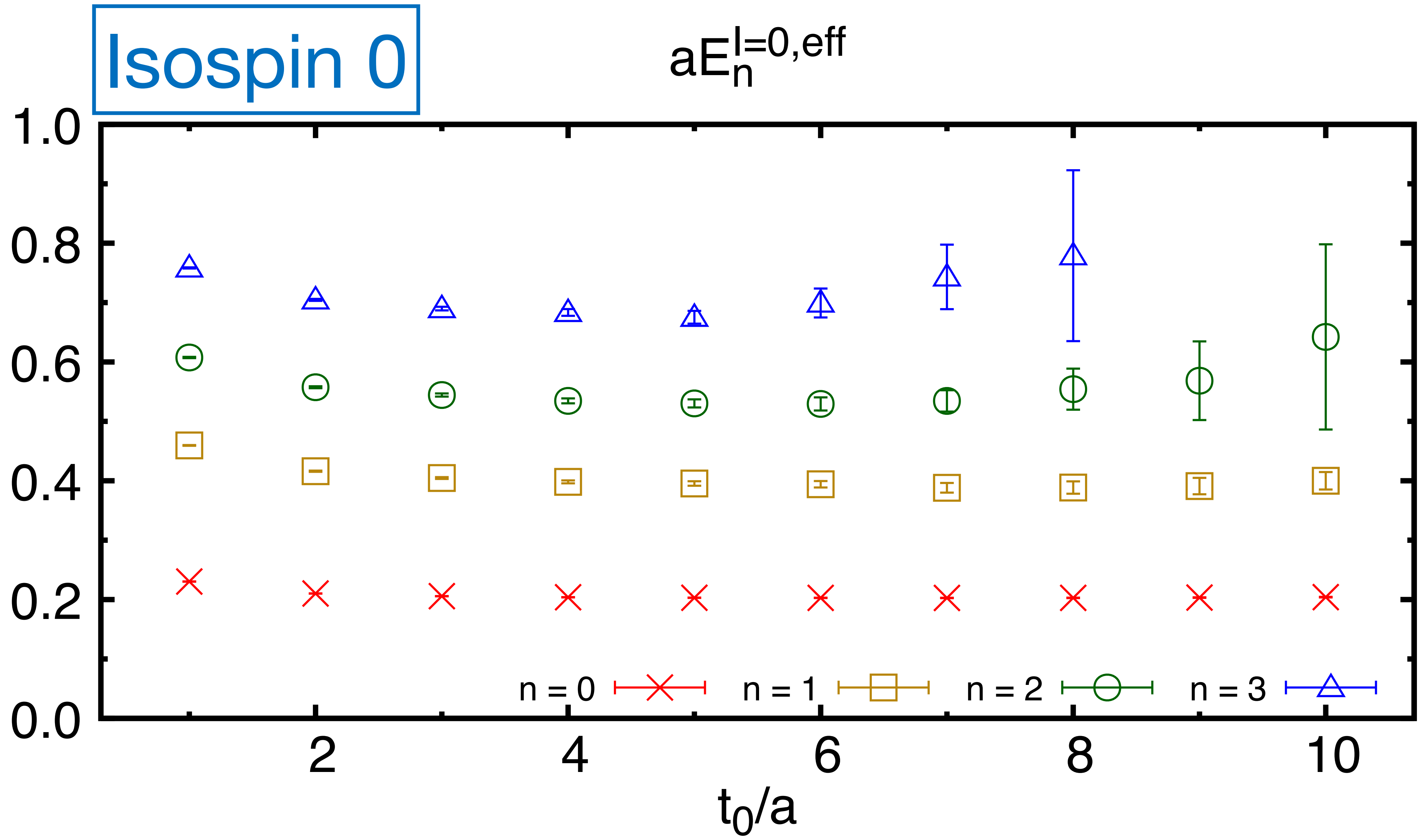
Isospin 2



ππ energies from GEVP

⟷

Isospin 0



What to calculate

- Single Kaon 2pt function

$$\langle O_K(t) O_K(0)^\dagger \rangle \to \langle 0 | O_K | 0 \rangle \langle 0 | O_K^\dagger | 0 \rangle e^{-m_K t}$$

- ππ 2pt functions

$$\langle O_a^{\pi\pi}(t) O_b^{\pi\pi}(0)^\dagger \rangle \xrightarrow{\text{GEVP}} v_{n,a},\ \langle 0 | O_n^{\pi\pi} | E_n^{\pi\pi} \rangle,\ E_n^{\pi\pi}$$

- K → ππ 3pt functions

$$\langle O_a^{\pi\pi}(t_1 + t_2) Q_i(t_1) O_K(0)^\dagger \rangle$$

$$\xrightarrow[\text{large } t_1 \text{ \& } t_2]{\sum_a v_{n,a} \times} \langle 0 | O_n^{\pi\pi} | E_n^{\pi\pi} \rangle \underline{\langle E_n^{\pi\pi} | Q_i | m_K \rangle} \langle m_K | O_K^\dagger | 0 \rangle e^{-E_n^{\pi\pi} t_2} e^{-m_K t_1}$$

ME with ππ excited state

What to calculate

- Single Kaon 2pt function

$$\left\langle O_K(t) O_K(0)^\dagger \right\rangle \to \left\langle 0 \middle| O_K \middle| 0 \right\rangle \left\langle 0 \middle| O_K^\dagger \middle| 0 \right\rangle e^{-m_K t}$$

- ππ 2pt functions

$$\left\langle O_a^{\pi\pi}(t) O_b^{\pi\pi}(0)^\dagger \right\rangle \xrightarrow{\text{GEVP}} v_{n,a},\ \left\langle 0 \middle| O_n^{\pi\pi} \middle| E_n^{\pi\pi} \right\rangle,\ E_n^{\pi\pi}$$

Today's topic
How to calculate these correlation functions

- K → ππ 3pt functions

$$\left\langle O_a^{\pi\pi}(t_1 + t_2) Q_i(t_1) O_K(0)^\dagger \right\rangle$$

ME with ππ excited state

$$\xrightarrow[\text{large } t_1 \,\&\, t_2]{\sum_a v_{n,a} \times} \left\langle 0 \middle| O_n^{\pi\pi} \middle| E_n^{\pi\pi} \right\rangle \underline{\left\langle E_n^{\pi\pi} \middle| Q_i \middle| m_K \right\rangle} \left\langle m_K \middle| O_K^\dagger \middle| 0 \right\rangle e^{-E_n^{\pi\pi} t_2} e^{-m_K t_1}$$

Calculation of correlation functions

How to calculate correlation functions

- Ensemble average → Monte Carlo method

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}\bar{q}\mathcal{D}q\mathcal{D}U\, \mathcal{O}(\bar{q}, q, U) e^{-S(\bar{q},q,U)}$$

$$\mathcal{O} = O_1(x_1)O_2(x_2)\ldots O_n(x_n)$$

Wick contraction, correlator with given background gauge field

$$= \frac{1}{Z} \int \mathcal{D}U\, \underline{\widehat{\mathcal{O}}(D(U)^{-1}, U)}\, \underline{\det D(U) e^{-S_G(U)}} \xrightarrow{\text{Monte Carlo sampling}} \frac{1}{N} \sum_{i=1}^{N} \widehat{\mathcal{O}}(D(U_i)^{-1}, U_i)$$

Integrating out fermion fields

Boltzmann weight

- Flow of computation
    - Ensemble generation : U_1, U_2, …, U_N based on the Boltzmann weight
    - Measurement: Calculate observable $\widehat{\mathcal{O}}(D(U_i)^{-1}, U_i)$ for each configuration U_i **[today's topic]**
    - Statistical analysis

Measurements

$$\widehat{\mathcal{O}}(D(U_i)^{-1}, U_i) = \mathcal{O}\left(\frac{\partial}{\partial\theta}, \frac{\partial}{\partial\bar{\theta}}, U_i\right) \exp\left[-\bar{\theta}D(U_i)^{-1}\theta\right]\Big|_{\theta,\bar{\theta}=0}$$

- $\theta, \bar{\theta}$: Grassmann variables with spin, color, flavor and position indices

space-time

- Example: 2pt function of π^+

$$O_1(x_1) = \bar{u}(x_1) i\gamma_5 d(x_1), \;\; O_2(x_2) = O_1(x_2)^\dagger = \bar{d}(x_2) i\gamma_5 u(x_2)$$

$$\mathcal{O} = \bar{u}(x_1) i\gamma_5 d(x_1) \bar{d}(x_2) i\gamma_5 u(x_2)$$

$$\mathcal{O} \to \widehat{\mathcal{O}} : \quad \bar{u}(x_1) \to \frac{\partial}{\partial\theta_{(u,x_1)}} \qquad d(x_1) \to \frac{\partial}{\partial\bar{\theta}_{(d,x_1)}} \qquad \bar{d}(x_2) \to \frac{\partial}{\partial\theta_{(d,x_2)}} \qquad u(x_2) \to \frac{\partial}{\partial\bar{\theta}_{(u,x_2)}}$$

$$\longrightarrow \widehat{\mathcal{O}}(D(U_i)^{-1}, U_i) = \frac{\partial}{\partial\theta_{(u,x_1)}} i\gamma_5 \frac{\partial}{\partial\bar{\theta}_{(d,x_1)}} \frac{\partial}{\partial\theta_{(d,x_2)}} i\gamma_5 \frac{\partial}{\partial\bar{\theta}_{(u,x_2)}} \exp\left[-\bar{\theta}D(U_i)^{-1}\theta\right]\Big|_{\theta,\bar{\theta}=0}$$

Hand exercise of π+ 2pt func

$$\widehat{\mathcal{O}}(D(U_i)^{-1}, U_i) = \frac{\partial}{\partial\theta_{(u,x_1)}} i\gamma_5 \frac{\partial}{\partial\bar{\theta}_{(d,x_1)}} \frac{\partial}{\partial\theta_{(d,x_2)}} i\gamma_5 \frac{\partial}{\partial\bar{\theta}_{(u,x_2)}} \exp\left[-\bar{\theta}D(U_i)^{-1}\theta\right]\Big|_{\theta,\bar{\theta}=0}$$

$$= \frac{1}{2} \frac{\partial}{\partial\theta_{(u,x_1)}} i\gamma_5 \frac{\partial}{\partial\bar{\theta}_{(d,x_1)}} \frac{\partial}{\partial\theta_{(d,x_2)}} i\gamma_5 \frac{\partial}{\partial\bar{\theta}_{(u,x_2)}} \bar{\theta}D^{-1}\theta \cdot \bar{\theta}D^{-1}\theta$$

$$= \frac{\partial}{\partial\theta_{(u,x_1)}} i\gamma_5 \frac{\partial}{\partial\bar{\theta}_{(d,x_1)}} \frac{\partial}{\partial\theta_{(d,x_2)}} i\gamma_5 \bar{\delta}_{(u,x_2)} D^{-1}\theta \cdot \bar{\theta}D^{-1}\theta$$

$$\bar{\delta}_{(u,x_2)} = \frac{\partial\bar{\theta}}{\partial\bar{\theta}_{(u,x_2)}}$$

$$= \frac{\partial}{\partial\theta_{(u,x_1)}} i\gamma_5 \frac{\partial}{\partial\bar{\theta}_{(d,x_1)}} \frac{\partial}{\partial\theta_{(d,x_2)}} \bar{\theta}D^{-1}\theta \cdot i\gamma_5 \bar{\delta}_{(u,x_2)} D^{-1}\theta$$

$$= -\frac{\partial}{\partial\theta_{(u,x_1)}} i\gamma_5 \frac{\partial}{\partial\bar{\theta}_{(d,x_1)}} \bar{\theta}D^{-1}\delta_{(d,x_2)} \cdot i\gamma_5 \bar{\delta}_{(u,x_2)} D^{-1}\theta$$

$$= -\frac{\partial}{\partial\theta_{(u,x_1)}} i\gamma_5 \bar{\delta}_{(d,x_1)} D^{-1}\delta_{(d,x_2)} \cdot i\gamma_5 \bar{\delta}_{(u,x_2)} D^{-1}\theta$$

$$= -\mathrm{Tr}\left[i\gamma_5 D^{-1}_{(d,x_1),(d,x_2)} i\gamma_5 D^{-1}_{(u,x_2),(u,x_1)}\right]$$

K^0 2pt func: u → s
simplest example

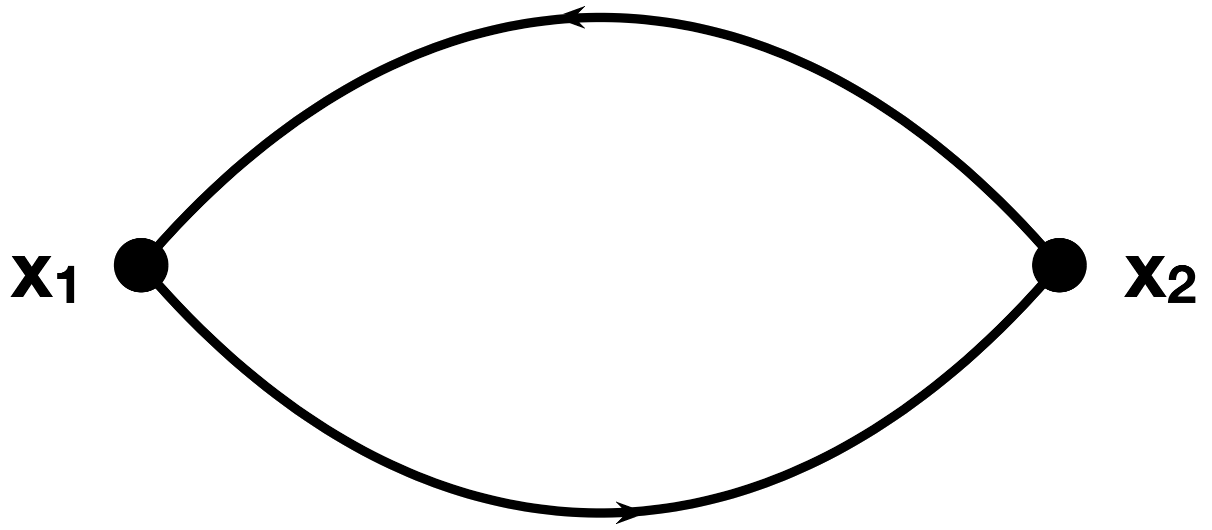
example 2: $\eta_{SU(2)}$

- source & sink operators

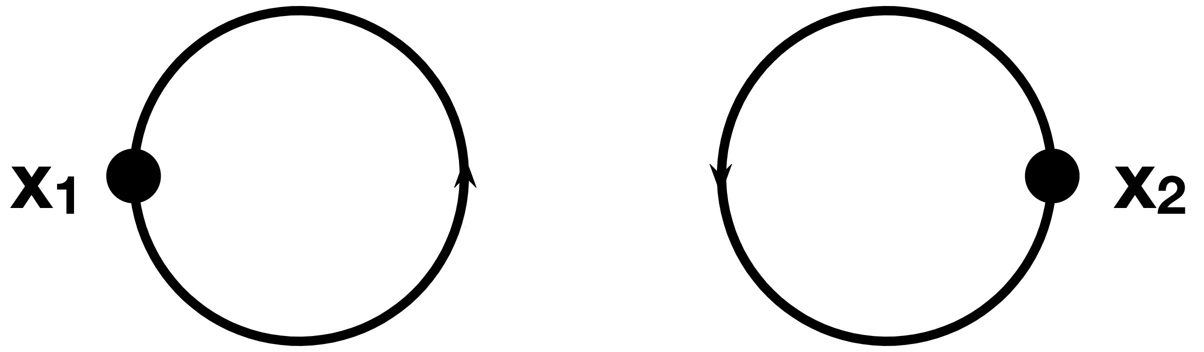
$$O_1(x_1) = \frac{\bar{q}(x_1) i\gamma_5 q(x_1)}{\sqrt{2}} \left(= \frac{\bar{u}(x_1) i\gamma_5 u(x_1) + \bar{d}(x_1) i\gamma_5 d(x_1)}{\sqrt{2}} \right), \ \ O_2(x_2) = O_1(x_2)^\dagger = \frac{\bar{q}(x_2) i\gamma_5 q(x_2)}{\sqrt{2}}$$

- contraction:

$$\mathrm{Tr}\left[\gamma_5 D^{-1}_{x_1,x_2} \gamma_5 D^{-1}_{x_2,x_1}\right] - \mathrm{Tr}\left[\gamma_5 D^{-1}_{x_1,x_1}\right] \mathrm{Tr}\left[\gamma_5 D^{-1}_{x_2,x_2}\right]$$



$\mathrm{Tr}\left[\gamma_5 D^{-1}_{x_1,x_2} \gamma_5 D^{-1}_{x_2,x_1}\right]$: connected diagram



$\mathrm{Tr}\left[\gamma_5 D^{-1}_{x_1,x_1}\right] \mathrm{Tr}\left[\gamma_5 D^{-1}_{x_2,x_2}\right]$: disconnected diagram

Automated Wick contractor for more complicated correlators

How should we obtain the contractions for ππ 2pt functions & K → ππ 3pt functions and compute them?

$\langle O^{\pi\pi}(x_1) O^{\pi\pi}(x_2)^\dagger \rangle$ $\qquad$ $\langle O^{\pi\pi}(x_1) Q_i(x_2) O_K(x_3)^\dagger \rangle$

$\langle O^{\pi\pi}(x_1) O_\sigma(x_2)^\dagger \rangle$ $\qquad$ $\langle O_\sigma(x_1) Q_i(x_2) O_K(x_3)^\dagger \rangle$

$$O^{\pi\pi} = \begin{cases} \dfrac{\pi^+\pi^- + 2\pi^0\pi^0 + \pi^-\pi^+}{\sqrt{6}} & (I = 2,\ I_z = 0) \\ \dfrac{\pi^+\pi^- - \pi^0\pi^0 + \pi^-\pi^+}{\sqrt{3}} & (I = 0) \end{cases}$$

$$Q_{1,2} = \bar{s}_\alpha \gamma_\mu (1-\gamma_5) u_{\beta/\alpha} \bar{u}_\beta \gamma_\mu (1-\gamma_5) d_{\alpha/\beta}$$

$$Q_{3,4,5,6} = \bar{s}_\alpha \gamma_\mu (1-\gamma_5) d_{\alpha/\beta} \sum_q \bar{q}_\beta \gamma_\mu (1 \pm \gamma_5) q_{\beta/\alpha}$$

$$Q_{7,8,9,10} = \bar{s}_\alpha \gamma_\mu (1-\gamma_5) d_{\alpha/\beta} \sum_q e_q \bar{q}_\beta \gamma_\mu (1 \pm \gamma_5) q_{\beta/\alpha}$$

done in earlier works but ...

PRD84,114503(2011)

$$A_{0,1}(t_\pi, t_{\rm op}, t_K) = i\frac{1}{\sqrt{3}}\{-\textcircled{1} - 2\cdot\textcircled{5} + 3\cdot\textcircled{9} + 3\cdot\textcircled{17} - 3\cdot\textcircled{33}\} \tag{6a}$$

$$A_{0,2}(t_\pi, t_{\rm op}, t_K) = i\frac{1}{\sqrt{3}}\{-\textcircled{2} - 2\cdot\textcircled{6} + 3\cdot\textcircled{10} + 3\cdot\textcircled{18} - 3\cdot\textcircled{34}\} \tag{6b}$$

$$A_{0,3}(t_\pi, t_{\rm op}, t_K) = i\sqrt{3}\{-\textcircled{5} + 2\cdot\textcircled{9} - \textcircled{13} + 2\cdot\textcircled{17} + \textcircled{21} - \textcircled{25} - \textcircled{29} - 2\cdot\textcircled{33} - \textcircled{37} + \textcircled{41} + \textcircled{45}\} \tag{6c}$$

$$A_{0,4}(t_\pi, t_{\rm op}, t_K) = i\sqrt{3}\{-\textcircled{6} + 2\cdot\textcircled{10} - \textcircled{14} + 2\cdot\textcircled{18} + \textcircled{22} - \textcircled{26} - \textcircled{30} - 2\cdot\textcircled{34} - \textcircled{38} + \textcircled{42} + \textcircled{46}\} \tag{6d}$$

$$A_{0,5}(t_\pi, t_{\rm op}, t_K) = i\sqrt{3}\{-\textcircled{7} + 2\cdot\textcircled{11} - \textcircled{15} + 2\cdot\textcircled{19} + \textcircled{23} - \textcircled{27} - \textcircled{31} - 2\cdot\textcircled{35} - \textcircled{39} + \textcircled{43} + \textcircled{47}\} \tag{6e}$$

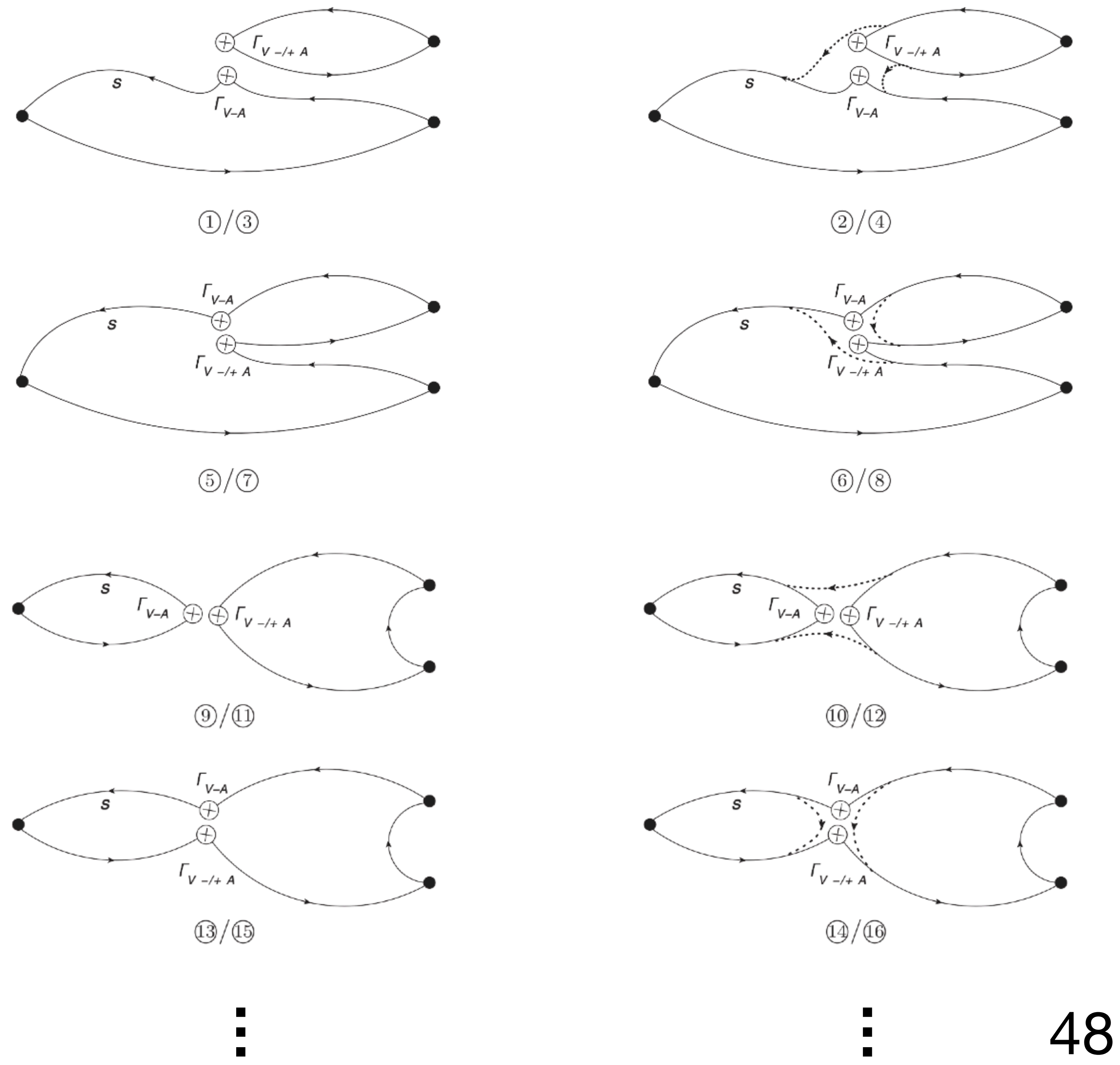
$$A_{0,6}(t_\pi, t_{\rm op}, t_K) = i\sqrt{3}\{-\textcircled{8} + 2\cdot\textcircled{12} - \textcircled{16} + 2\cdot\textcircled{20} + \textcircled{24} - \textcircled{28} - \textcircled{32} - 2\cdot\underset{??}{\textcircled{32}} - \textcircled{40} + \textcircled{44} + \textcircled{48}\} \tag{6f}$$

$$A_{0,7}(t_\pi, t_{\rm op}, t_K) = i\frac{\sqrt{3}}{2}\{-\textcircled{3} - \textcircled{7} + \textcircled{11} + \textcircled{15} + \textcircled{19} - \textcircled{23} + \textcircled{27} + \textcircled{31} - \textcircled{35} + \textcircled{39} - \textcircled{43} - \textcircled{47}\} \tag{6g}$$

$$A_{0,8}(t_\pi, t_{\rm op}, t_K) = i\frac{\sqrt{3}}{2}\{-\textcircled{4} - \textcircled{8} + \textcircled{12} + \textcircled{16} + \textcircled{20} - \textcircled{24} + \textcircled{28} + \underset{??}{\textcircled{38}} - \textcircled{32} + \textcircled{40} - \textcircled{44} - \textcircled{48}\} \tag{6h}$$

$$A_{0,9}(t_\pi, t_{\rm op}, t_K) = i\frac{\sqrt{3}}{2}\{-\textcircled{1} - \textcircled{5} + \textcircled{9} + \textcircled{13} + -\textcircled{21} + \textcircled{25} + \textcircled{29} - \textcircled{33} + \textcircled{37} - \textcircled{41} - \textcircled{45}\} \tag{6i}$$

$$A_{0,10}(t_\pi, t_{\rm op}, t_K) = i\frac{\sqrt{3}}{2}\{-\textcircled{2} - \textcircled{6} + \textcircled{10} + \textcircled{14} + \textcircled{18} - \textcircled{22} + \textcircled{26} + \textcircled{30} - \textcircled{34} + \textcircled{38} - \textcircled{42} - \textcircled{46}\}, \tag{6j}$$



48 contractions

- easy to make typos in hand coding (just as the PRD proofreader did)
- another headache to ensure about normalization
- K → σ contraction only in GPBC work (where contractions slightly differ)

Automated Wick contractor

- Input: a set of operators
- Output: Wick contractions for the correlation function of the given operators

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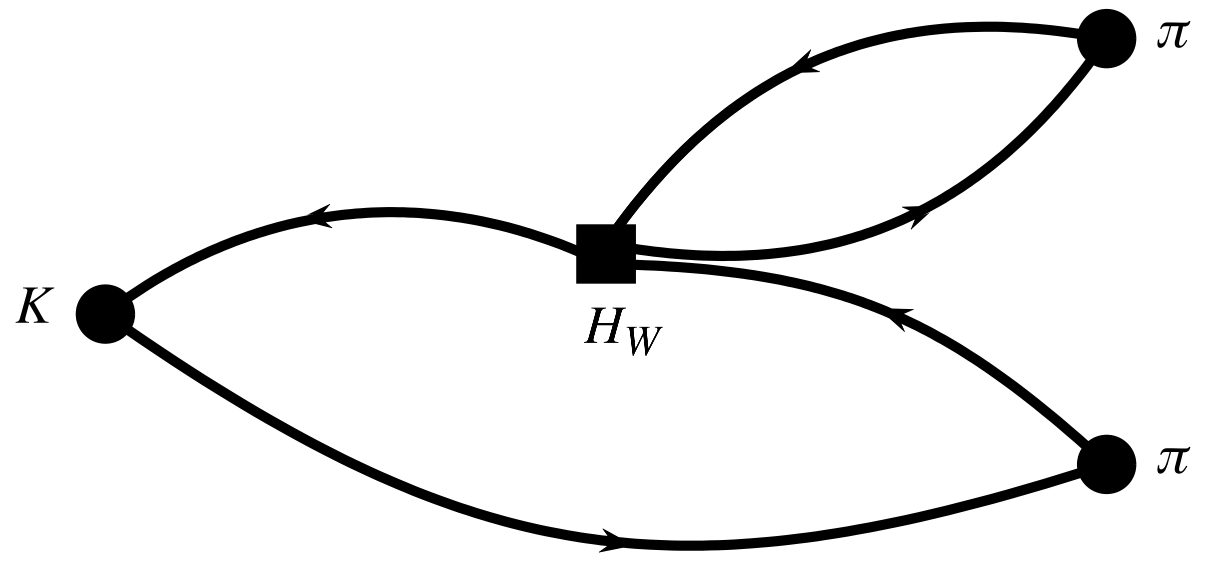
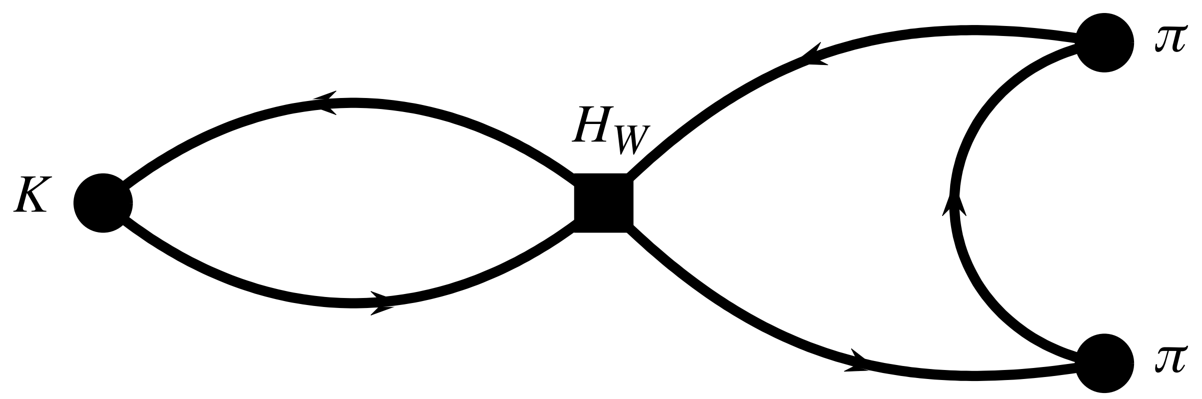
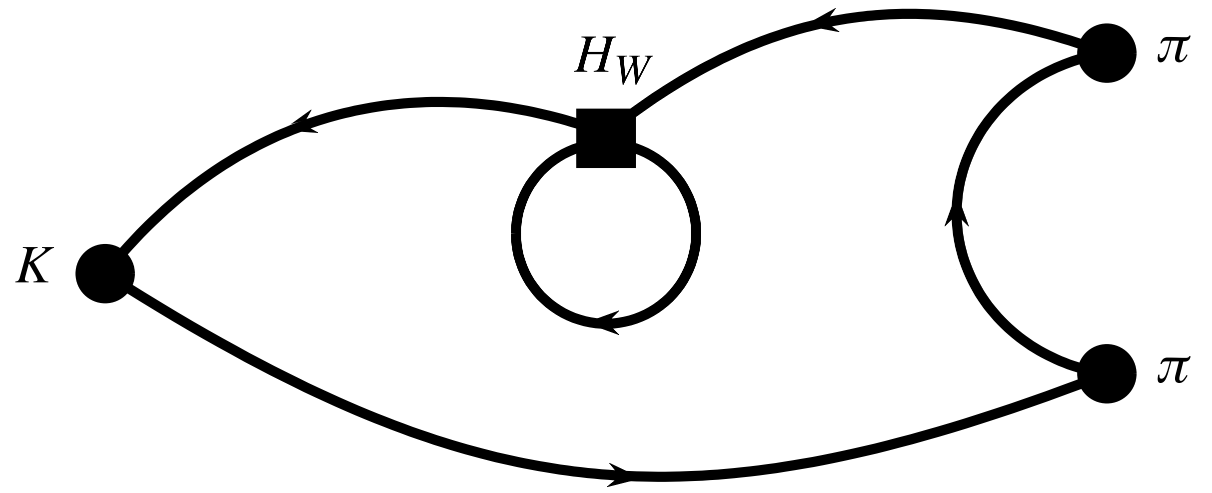


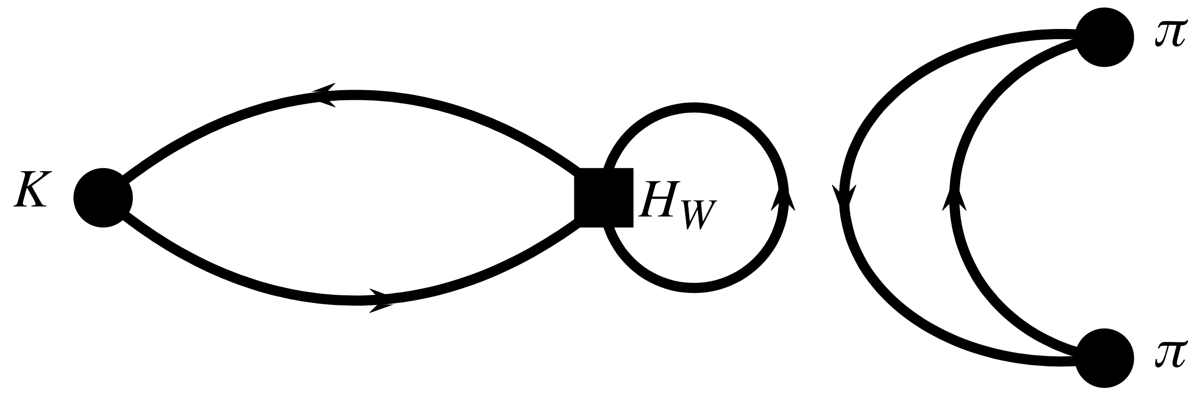
type 1



type 2



type 3



type 4

Automated Wick contractor

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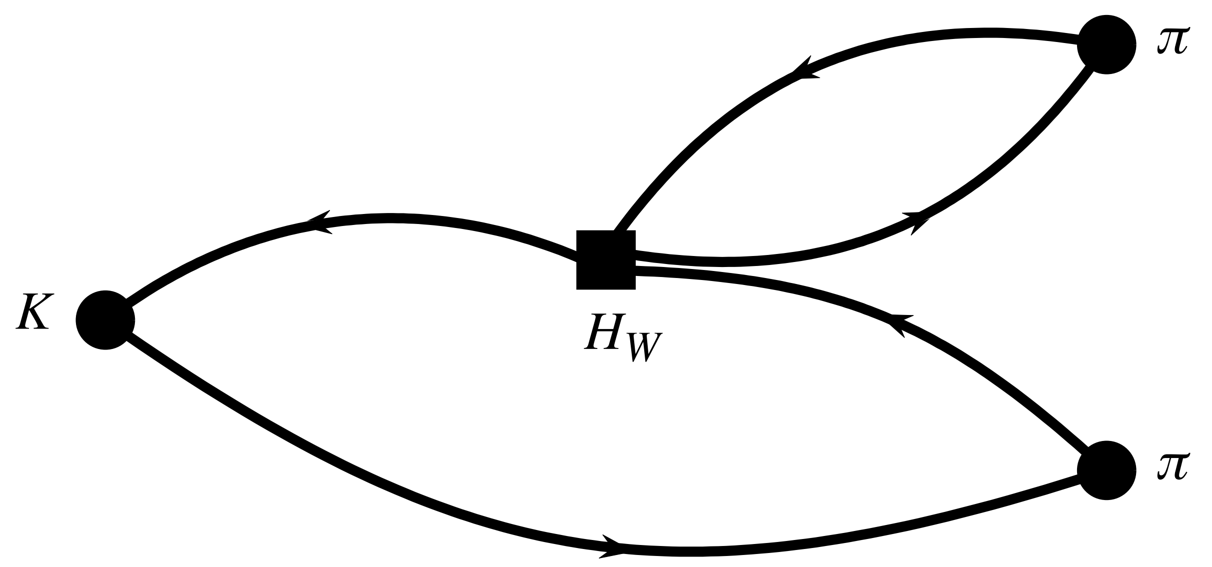

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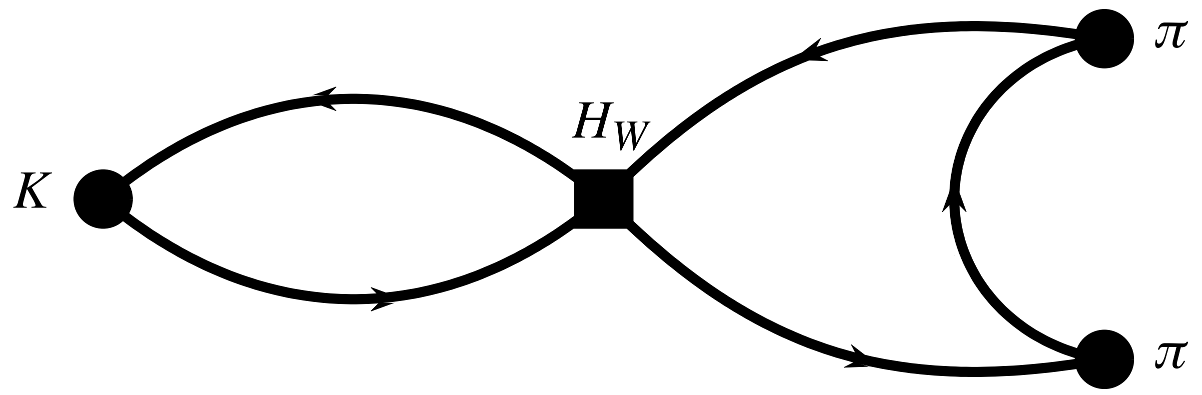

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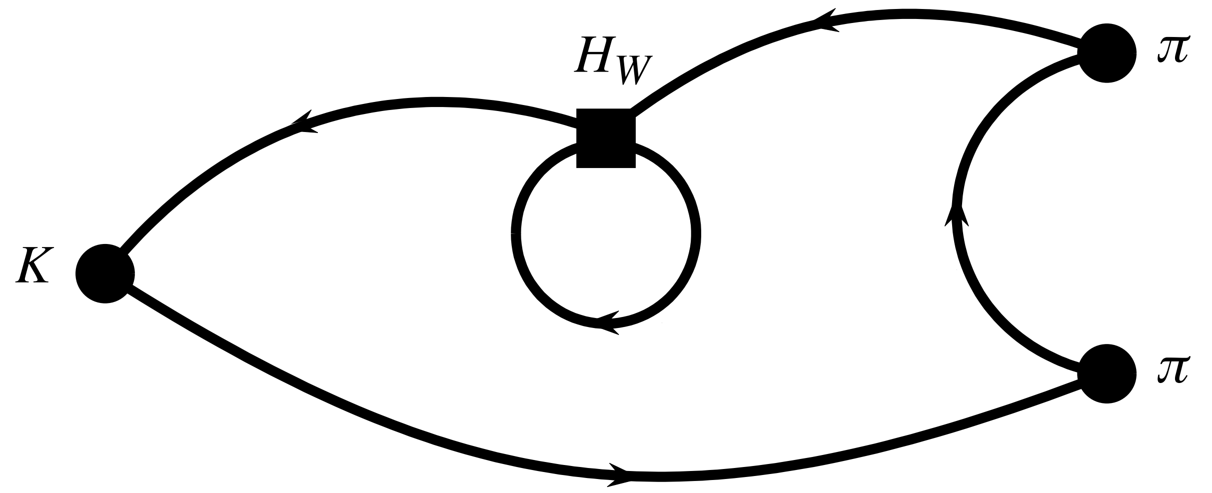
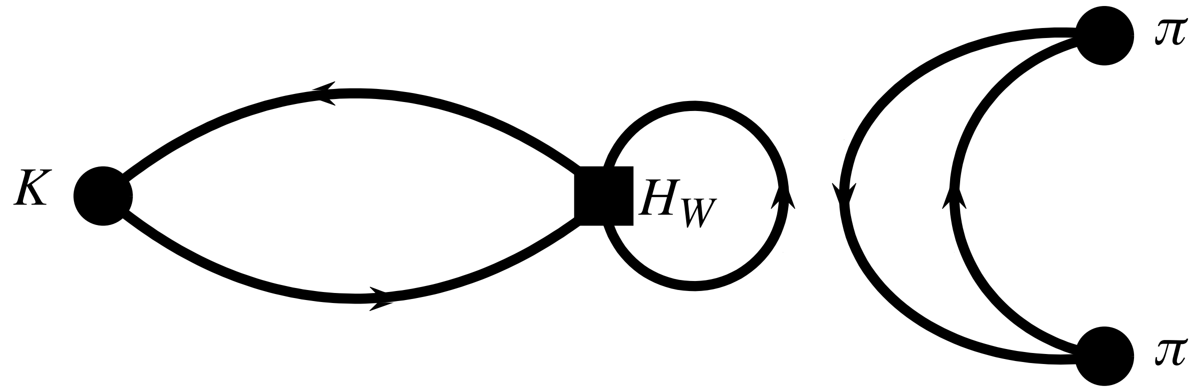


type 3



type 4

Let the script recognize these diagrams and rename contractions

Automated Wick contractor

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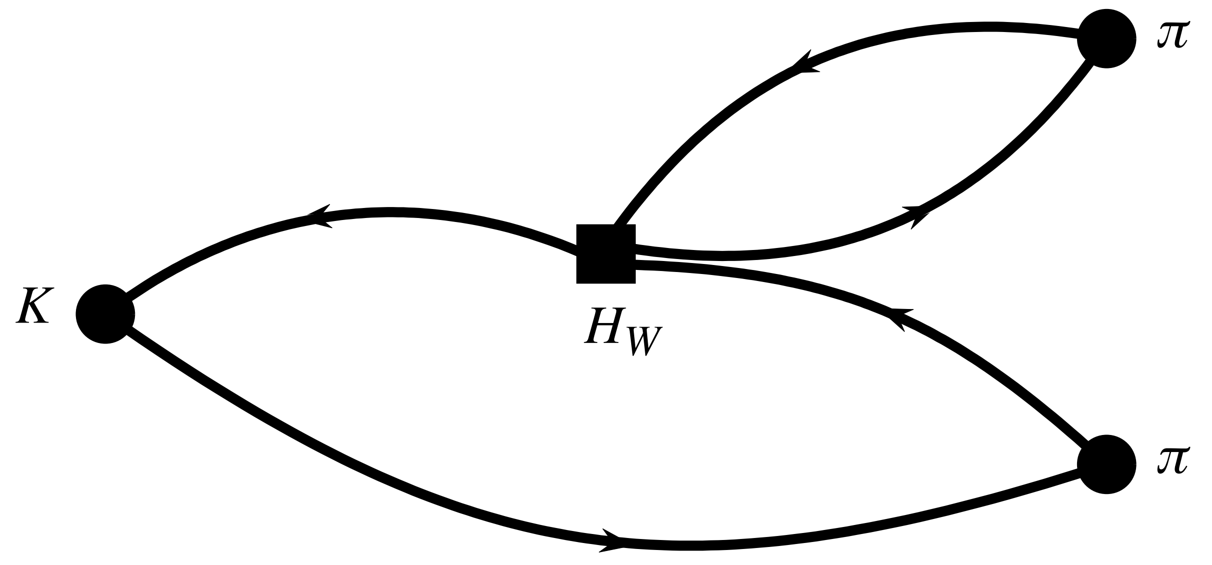
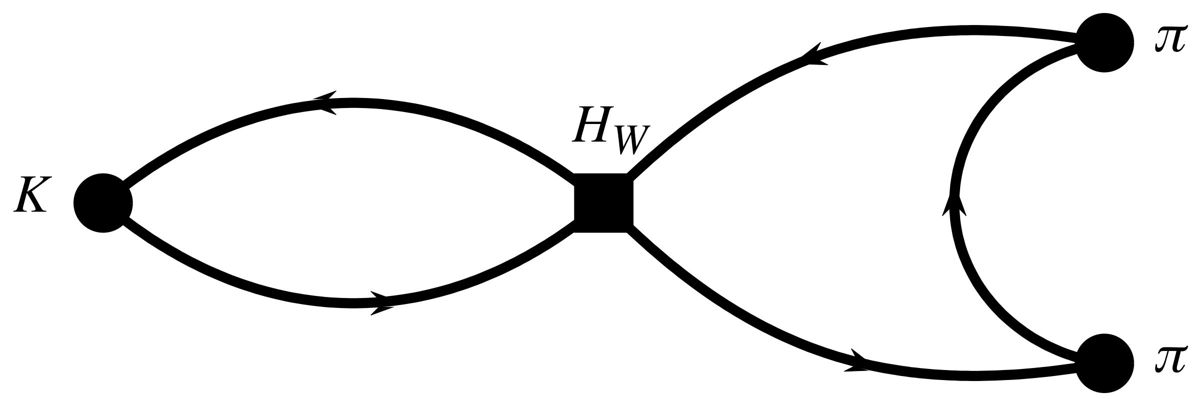
- Output: Wick contractions for the correlation function of the given operators



type 1



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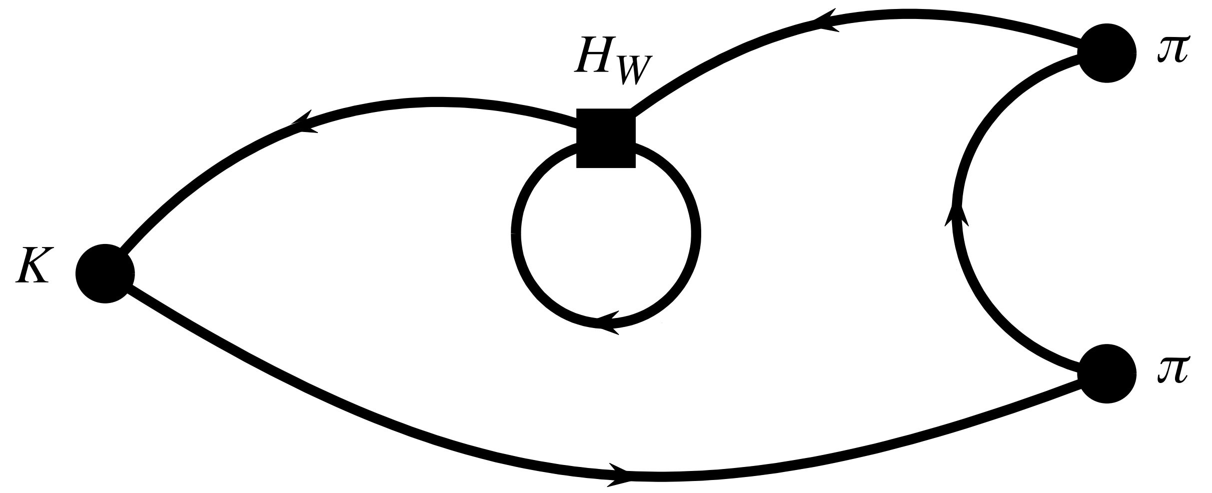

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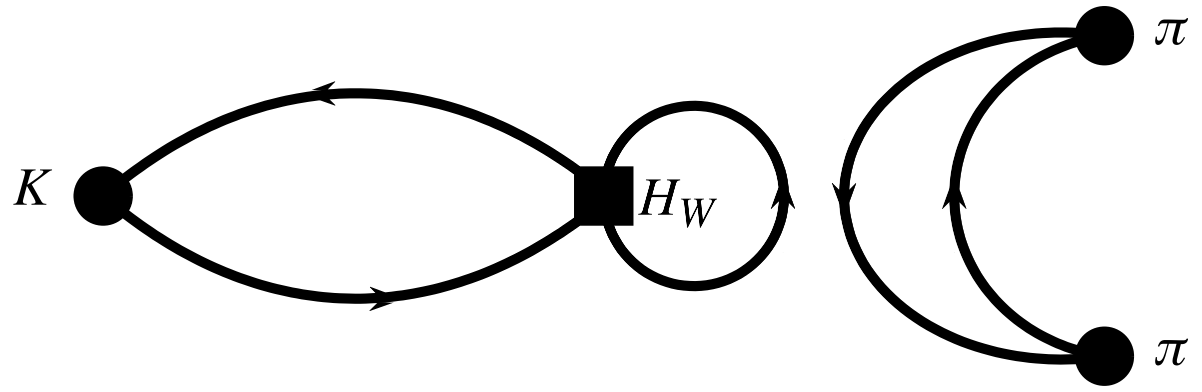

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```
-3/2 type1_singleTr_cmix_LR
+3 type2_multiTr_cmix_LR
-3/2 type2_singleTr_cmix_LR
+3 type3_lloop_multiTr_cmix_LR
-3/2 type3_lloop_singleTr_cmix_LR
+3/2 type3_sloop_multiTr_cmix_LR
-3/2 type3_sloop_singleTr_cmix_RL
-6 type4_lloop_multiTr_cmix_LR
+3 type4_lloop_singleTr_cmix_LR
-3 type4_sloop_multiTr_cmix_LR
+3 type4_sloop_singleTr_cmix_RL
```

example: $\langle O^{\pi\pi,I=0}(x_1)Q_6(x_2)O_K(x_3)^\dagger\rangle$

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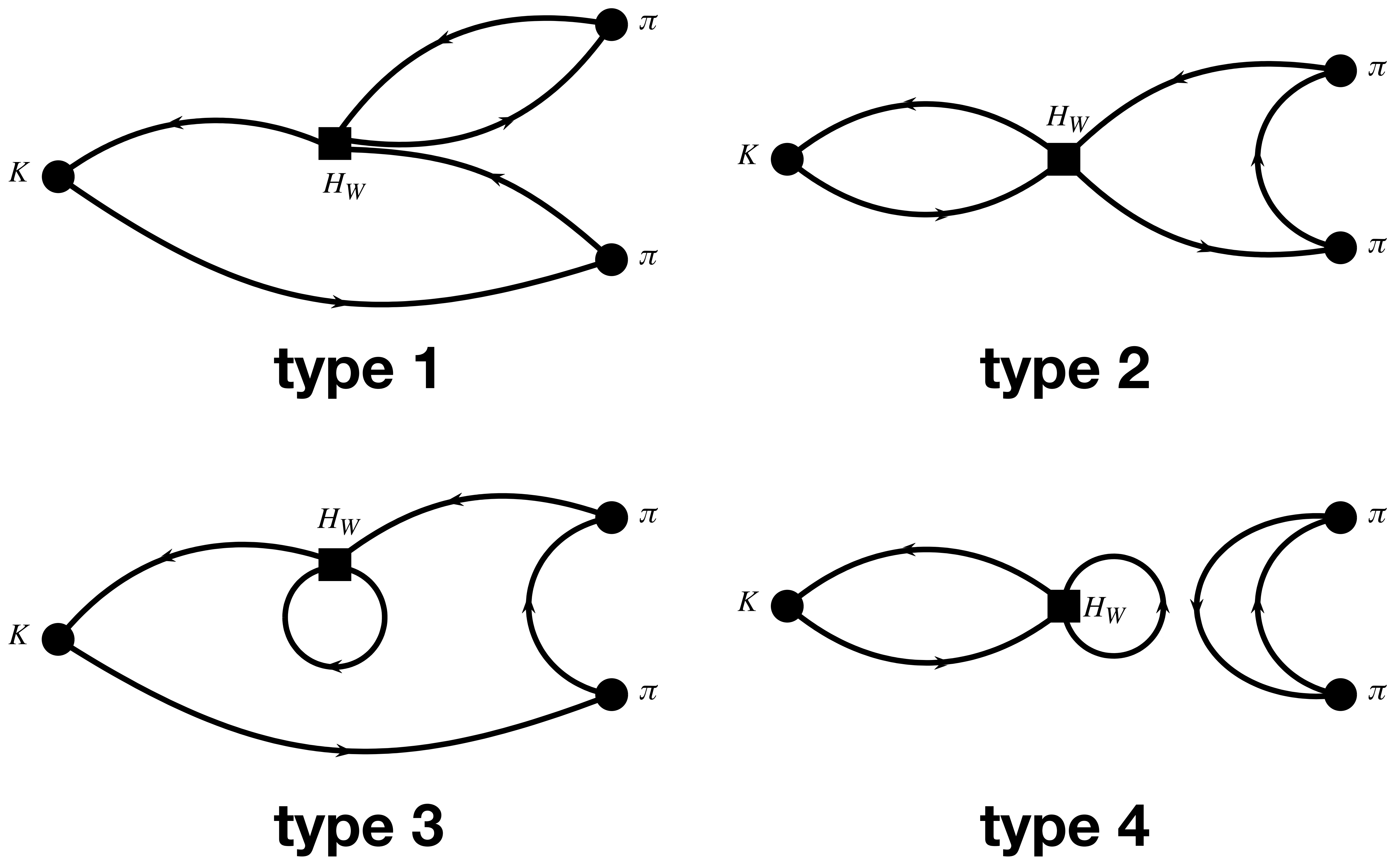

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+3/2 type3_sloop_multiTr_cmix_LR
-3/2 type3_sloop_singleTr_cmix_RL
-6 type4_lloop_multiTr_cmix_LR
+3 type4_lloop_singleTr_cmix_LR
-3 type4_sloop_multiTr_cmix_LR
+3 type4_sloop_singleTr_cmix_RL
```

example: $\langle O^{\pi\pi,I=0}(x_1)Q_6(x_2)O_K(x_3)^\dagger\rangle$

compute each contraction, and let analysis code do the linear combination

Summary so far and what's next

- We now know how to combine each contraction to calculate correlation functions
- Need to figure out
  - spatial momentum projection
  - how to precisely compute these momentum-projected contractions

Detailed strategy for $K \to \pi\pi$

Momentum projection

- Kaon operator: zero spatial momentum (rest frame)

$$O_K(t)^\dagger = \sum_{\vec{x}} \bar{d}(\vec{x},t) i\gamma_5 s(\vec{x},t)\; e^{i\vec{0}\cdot\vec{x}}$$

- ππ operators made with back-to-back momenta

$$O^{\pi\pi,I}_{\vec{p}}(t)^\dagger = \underline{\sum_{a,b}} c^I_{ab} \underline{\sum_{\vec{p}'\in\hat{T}[\vec{p}]}} \sum_{\vec{x}} e^{i\vec{p}'\cdot x} \pi^a(\vec{x},t) \sum_{\vec{y}} e^{-i\vec{p}'\cdot y} \pi^b(\vec{y},t+\Delta)$$

Sum over 3D cubic rotation: s-wave projection

Isospin projection taken into account by auto contractor

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Sum over 3D cubic rotation: s-wave projection

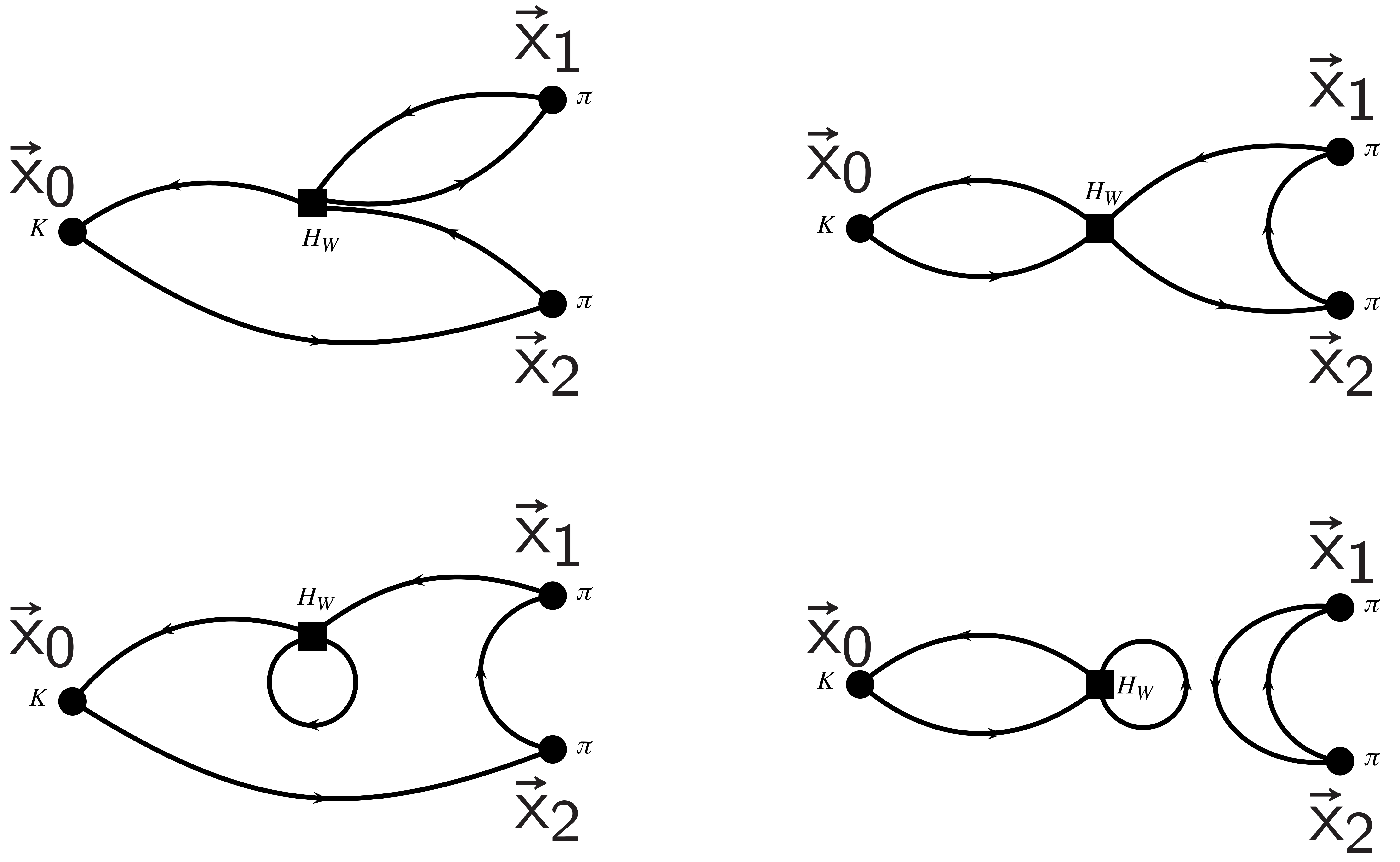
Isospin projection taken into account by auto contractor

presence of spatial sums

How contractions look

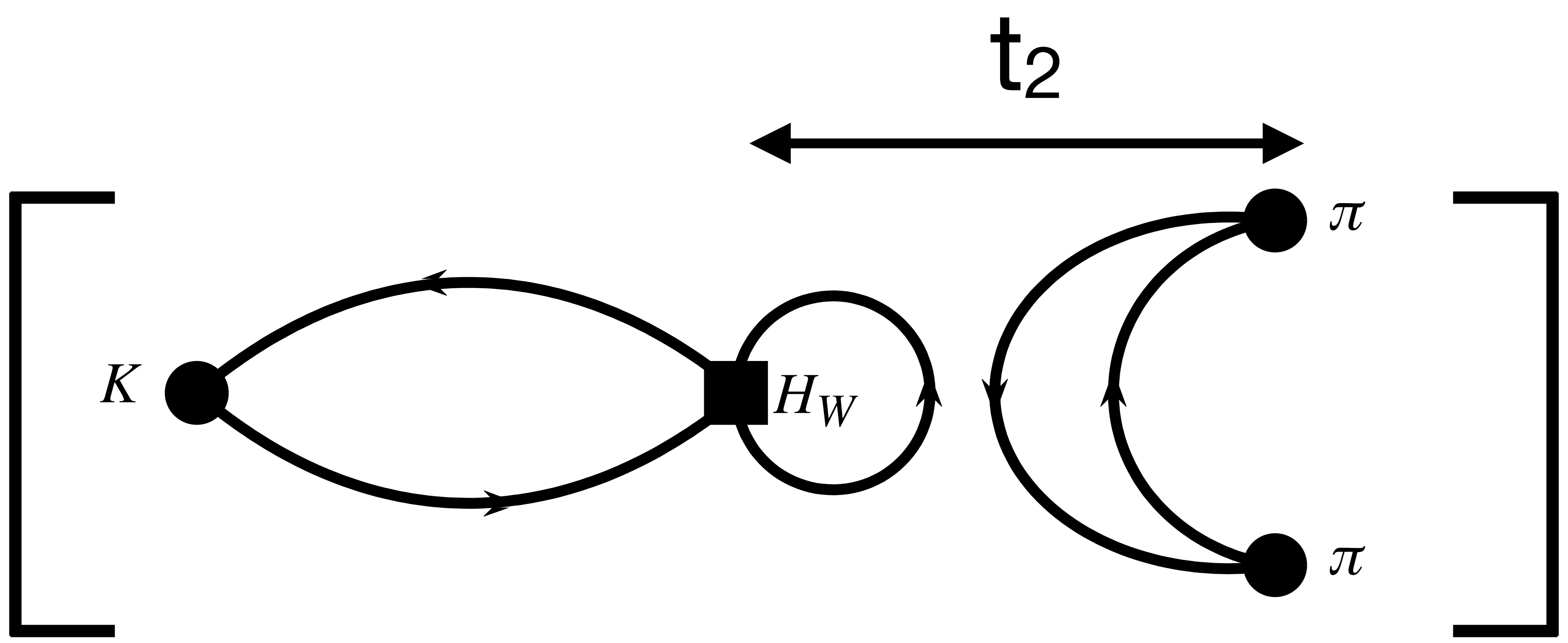
$$\sum_{\vec{x}_0} \sum_{\vec{x}_1} e^{i\vec{p}' \cdot \vec{x}_1} \sum_{\vec{x}_2} e^{-i\vec{p}' \cdot \vec{x}_2}$$

- minimal positional sums presumed
- adding other sums wrt translational invariance may help statistics



Disconnected diagram noisiest

S/N



$\sim e^{-E_n^{\pi\pi} t_2} / 1$

S/N [other diagrams] $\sim e^{-E_n^{\pi\pi} t_2} / e^{-2m_\pi t_2}$

- Natural thing to consider: maximize effective sampling
- In this work
  - averaging over all spacetime translations of H_W for disconnected diagram
  - don't need to sample other diagrams as many if it's expensive

Calculation of quark propagators

Lattice calculation of quark propagators

- Not possible to get the exact inverse of Dirac operator in large volumes
- What we can do: solve linear equations of Dirac operator
  - point-source propagator

$$\sum_y D(x,y)X(y) = \delta_{x,x_0} \rightarrow X(x) = D^{-1}(x,x_0)$$

spin, color (and flavor) indices suppressed

  - momentum-source propagator

$$\sum_y D(x,y)X(y) = e^{ipx} \rightarrow X(x) = \sum_y e^{ipy} D^{-1}(x,y)$$

4D momentum

$$\sum_y D(x,y)X(y) = e^{i\vec{p}\cdot\vec{x}}\delta_{x_4,t_0} \rightarrow X(x) = \sum_{\vec{y}} e^{i\vec{p}\cdot\vec{y}} D^{-1}(x,(\vec{y},t_0))$$

3D momentum

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3D momentum

These were considered unrealistic for the disconnected diagram

all-to-all (A2A) propagator method

- method to calculate quark propagator connecting all sources to all sinks
- useful particularly (but not only) when a quark loop present in contraction diagrams
- Low-mode decomposition
  - Dirac operator $\gamma_5 D(x,y) = \sum_i v_i(x)\lambda_i v_i(y)^\dagger$

    spin, color (and flavor) indices suppressed
  - Quark propagator $S_F(x,y) = D^{-1}(x,y) = \sum_i v_i(x)\frac{1}{\lambda_i}v_i(y)^\dagger\gamma_5$

    ↑ truncated at $i = N_\ell \ll V$

    good approximation for light quarks
- Noise method for strange-quark propagator & light-quark high modes (next slide)

Noise method

if we find vectors $\xi_i(x)$ that satisfy

$$\sum_{i=1}^{N_h} \xi_i(x)\xi_i(y)^\dagger \approx \delta_{x,y}$$

solve

$$\sum_y D(x,y)X_i(y) = \xi_i(x) \rightarrow X_i(x) = \sum_y D^{-1}(x,y)\xi_i(y)$$

$$\rightarrow \sum_i X_i(x)\xi_i(y)^\dagger \approx D^{-1}(x,y)$$

Pure noise method used for strange-quark propagator

Noise method

if we find vectors $\xi_i(x)$ that satisfy

$$\sum_{i=1}^{N_h} \xi_i(x)\xi_i(y)^\dagger \approx \delta_{x,y}$$

$\xi_i(x)$: Random values in U(1) / $\sqrt{N_h}$ or its subgroup

$O(1/\sqrt{N_h})$ error taken into account as statistical error

solve

$$\sum_y D(x,y)X_i(y) = \xi_i(x) \rightarrow X_i(x) = \sum_y D^{-1}(x,y)\xi_i(y)$$

$$\rightarrow \sum_i X_i(x)\xi_i(y)^\dagger \approx D^{-1}(x,y)$$

Pure noise method used for strange-quark propagator

Combining low-mode approximation & noise method

$$D^{-1}(x,y) \approx \sum_{i=1}^{N_\ell} v_i(x)\frac{1}{\lambda_i}v_i(y)^\dagger\gamma_5 + \sum_{j=1}^{N_h}\left(X_j(x) - \sum_{i=1}^{N_\ell}\sum_z v_i(x)\frac{1}{\lambda_i}v_i(z)^\dagger\gamma_5\xi_j(z)\right)\xi_j(y)^\dagger$$

low- & high-mode decomposition

$$D^{-1}(x,y) \approx \sum_{i=1}^{N_\ell+N_h} V_i(x)W_i(y)^\dagger$$

$O(10^9 \times 10^9)$ matrix → $O(10^3)$ vectors of size $O(10^8)$

lost information is not important for long-distance physics

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low- & high-mode decomposition

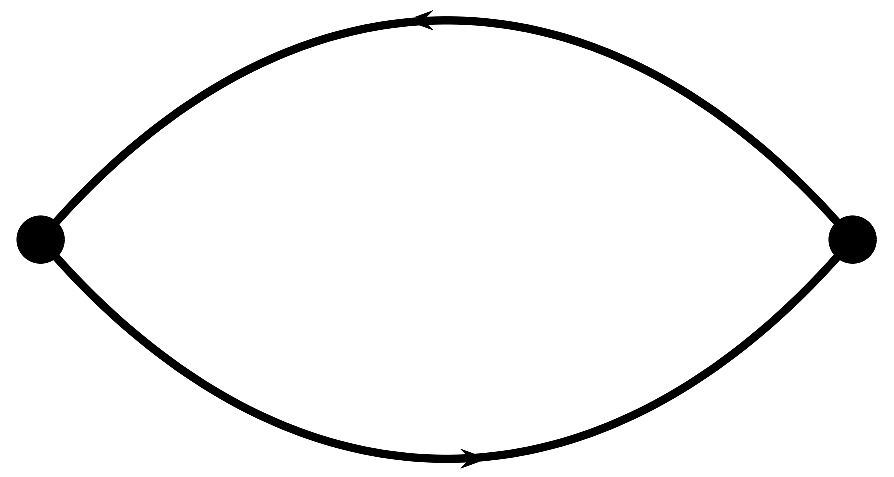
$$D^{-1}(x,y) \approx \sum_{i=1}^{N_\ell+N_h} V_i(x) W_i(y)^\dagger$$

10^{11} complex numbers → O(1) TB
good to consider for serious calculations

$O(10^9 \times 10^9)$ matrix → $O(10^3)$ vectors of size $O(10^8)$
lost information is not important for long-distance physics

A2A propagators for simplest contraction

- cannot do a huge "for loop" over V²
- disentangle and find a way for the contraction



$$\sum_{\vec{x},\vec{y}} \mathrm{Tr}[\gamma_5 D^{-1}((\vec{x},x_4),(\vec{y},y_4))\gamma_5 D^{-1}((\vec{y},y_4),(\vec{x},x_4))]$$

$$= \sum_{\vec{x},\vec{y}} \sum_{i,j} \mathrm{Tr}[\gamma_5 V_i(\vec{x},x_4) W_i(\vec{y},y_4)^\dagger \gamma_5 V_j(\vec{y},y_4) W_j(\vec{x},x_4)^\dagger]$$

$$= \sum_{i,j} \underbrace{\sum_{\vec{x}} W_j(\vec{x},x_4)^\dagger \gamma_5 V_i(\vec{x},x_4)}_{\Pi_{ij}^{\gamma_5}(x_4)} \cdot \underbrace{\sum_{\vec{y}} W_i(\vec{y},y_4)^\dagger \gamma_5 V_j(\vec{y},y_4)}_{\Pi_{ji}^{\gamma_5}(y_4)}$$

$$= \sum_{i,j} \Pi_{ij}^{\gamma_5}(x_4) \Pi_{ji}^{\gamma_5}(y_4)$$

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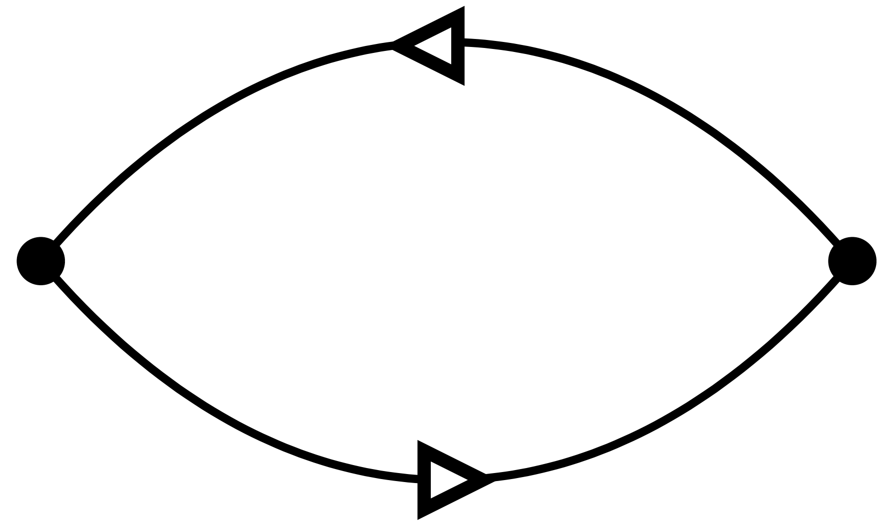

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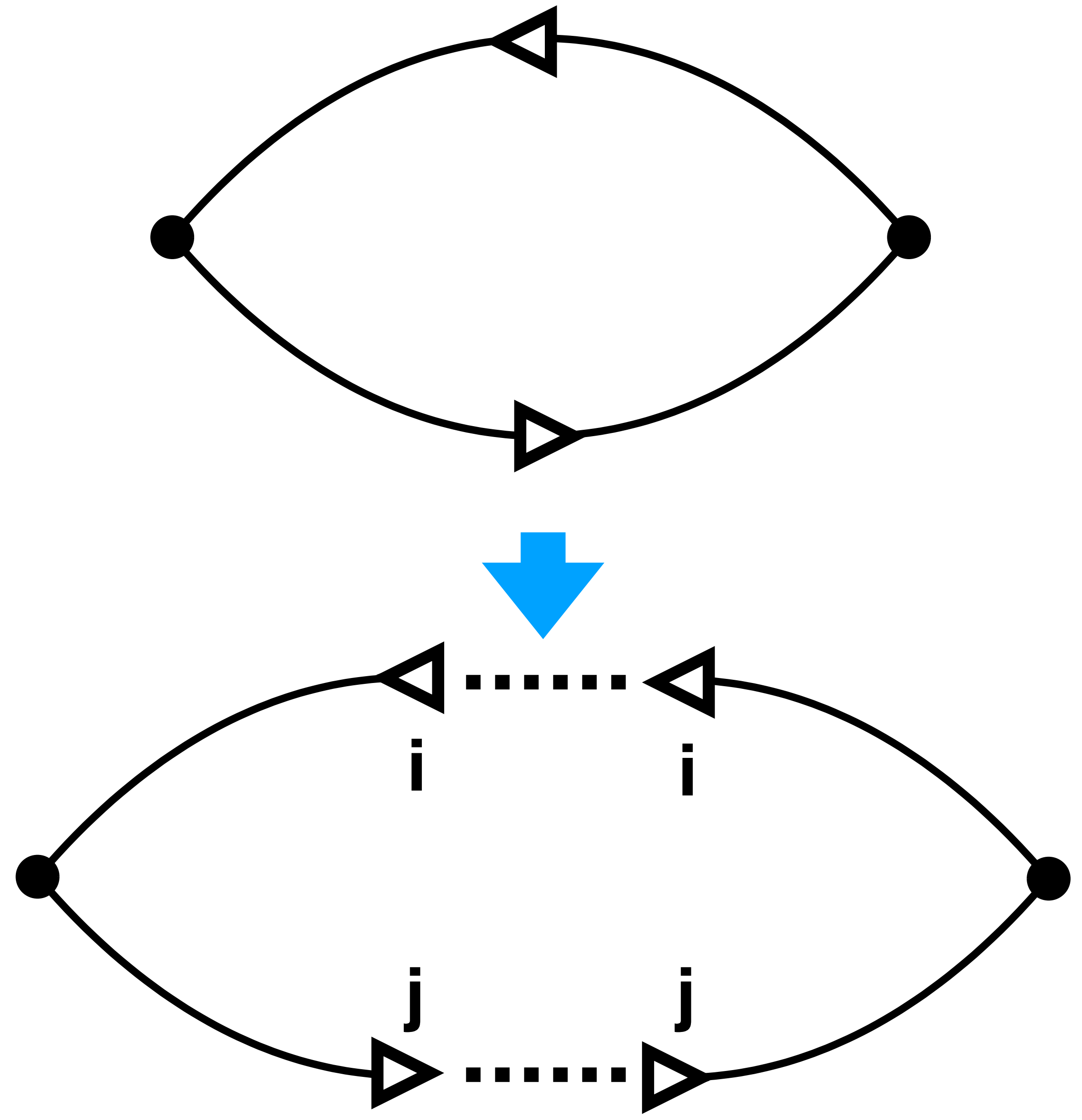

A2A propagators for simplest contraction

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$$\sum_{\vec{x},\vec{y}} \mathrm{Tr}[\gamma_5 D^{-1}((\vec{x},x_4),(\vec{y},y_4))\gamma_5 D^{-1}((\vec{y},y_4),(\vec{x},x_4))]$$

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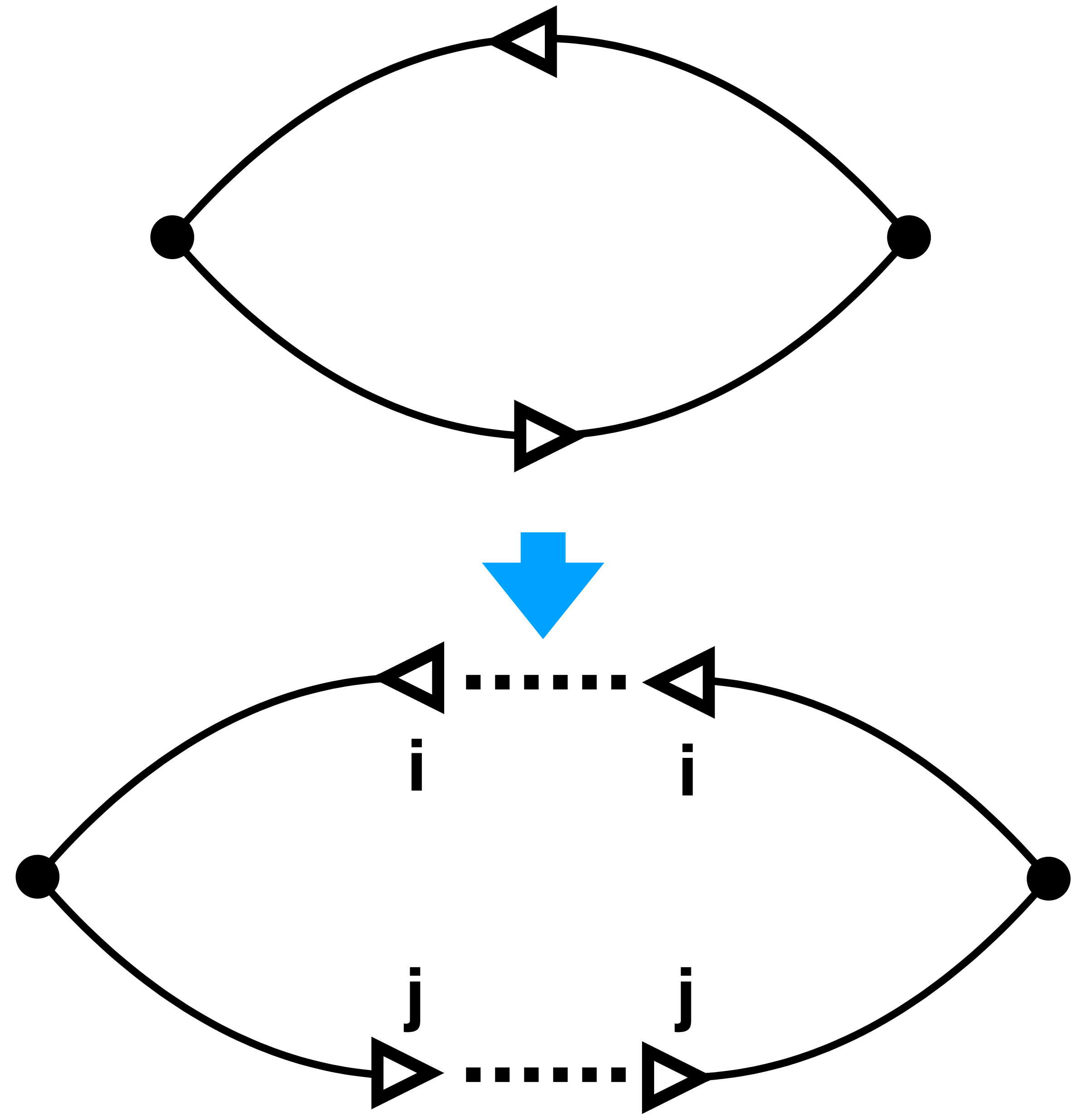
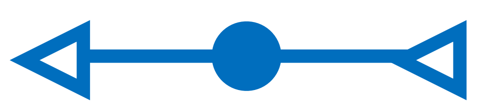


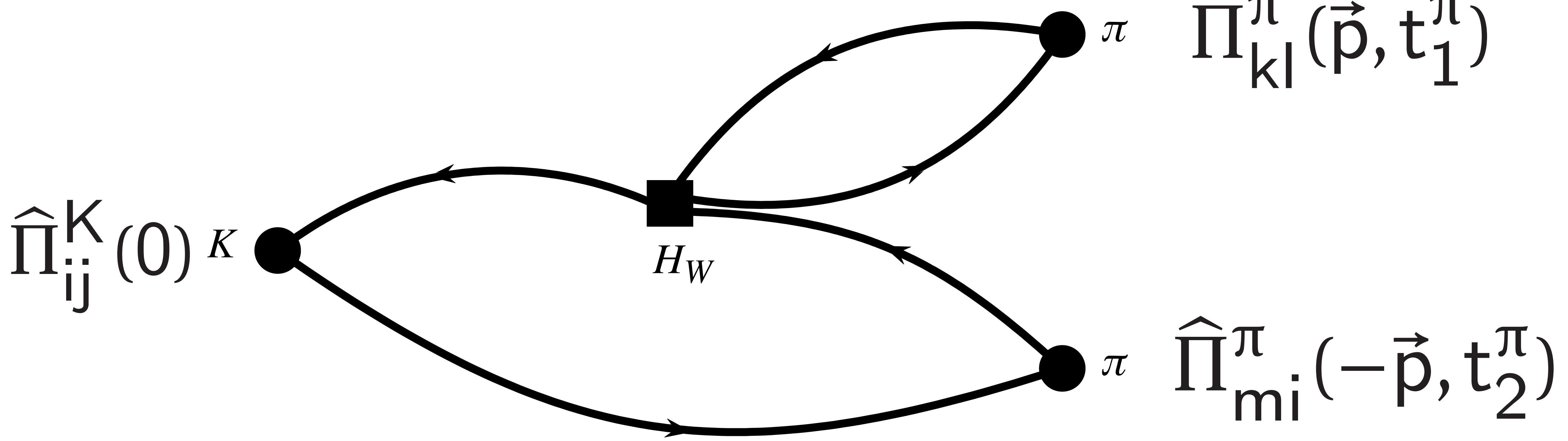


- A2A Meson field
- O(10^3x10^3x10^2) complex numbers → O(1) GB
- should be saved for other projects

Meson fields for K → ππ

$$\widehat{\Pi}^{K}_{ij}(t) = \sum_{\vec{x}} W^{\ell}_{i}(\vec{x},t) W^{s}_{j}(\vec{x},t)$$

$$\widehat{\Pi}^{\pi}_{ij}(\vec{p},t) = \sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}} W^{\ell}_{i}(\vec{x},t)\gamma_5 V^{\ell}_{j}(\vec{x},t)$$



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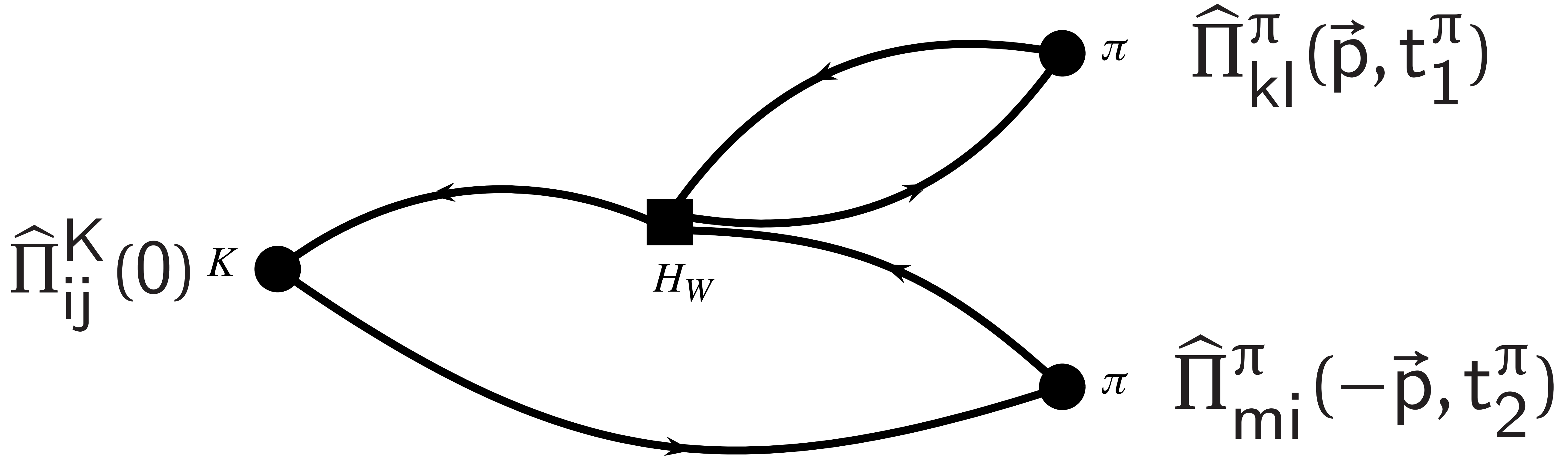

Only 4-quark operator remain with spin, color (& position) indices uncontracted

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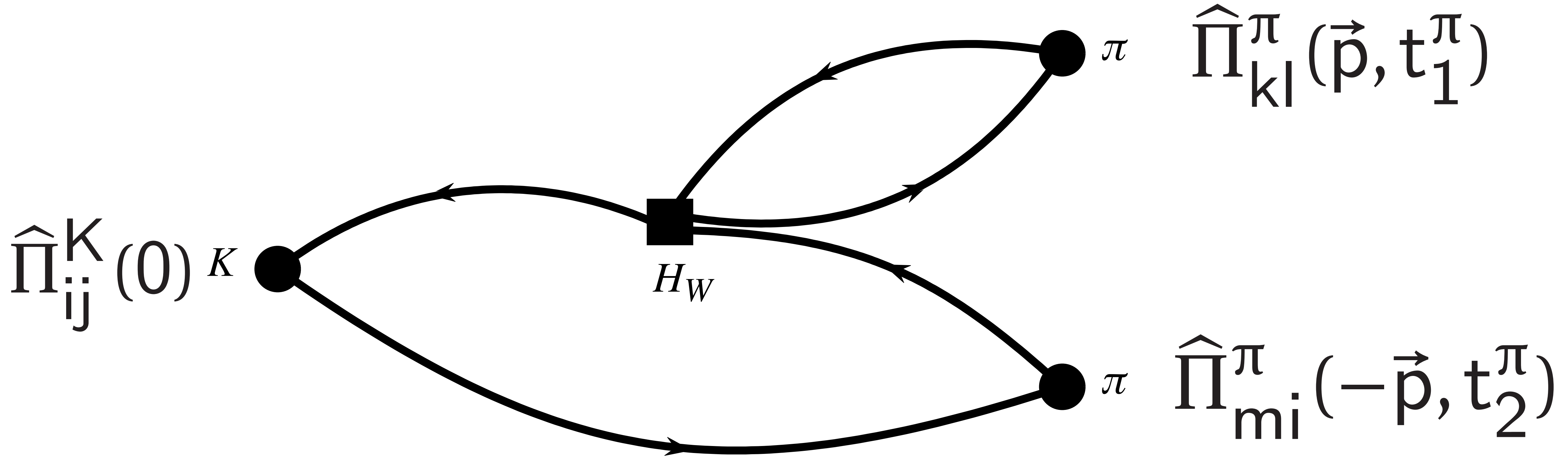
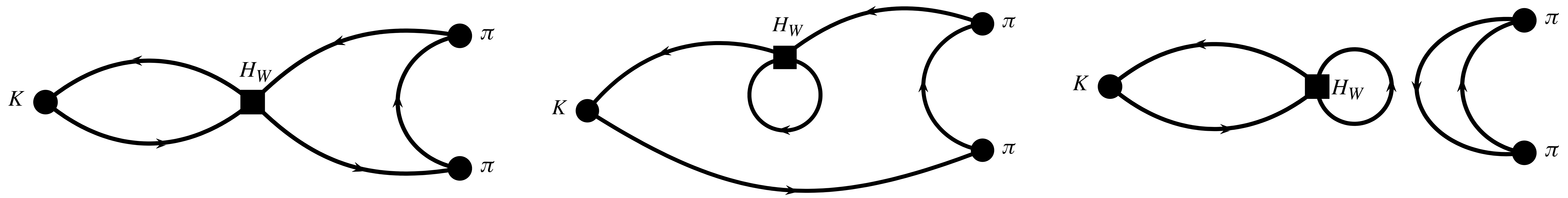


Only 4-quark operator remain with spin, color (& position) indices uncontracted

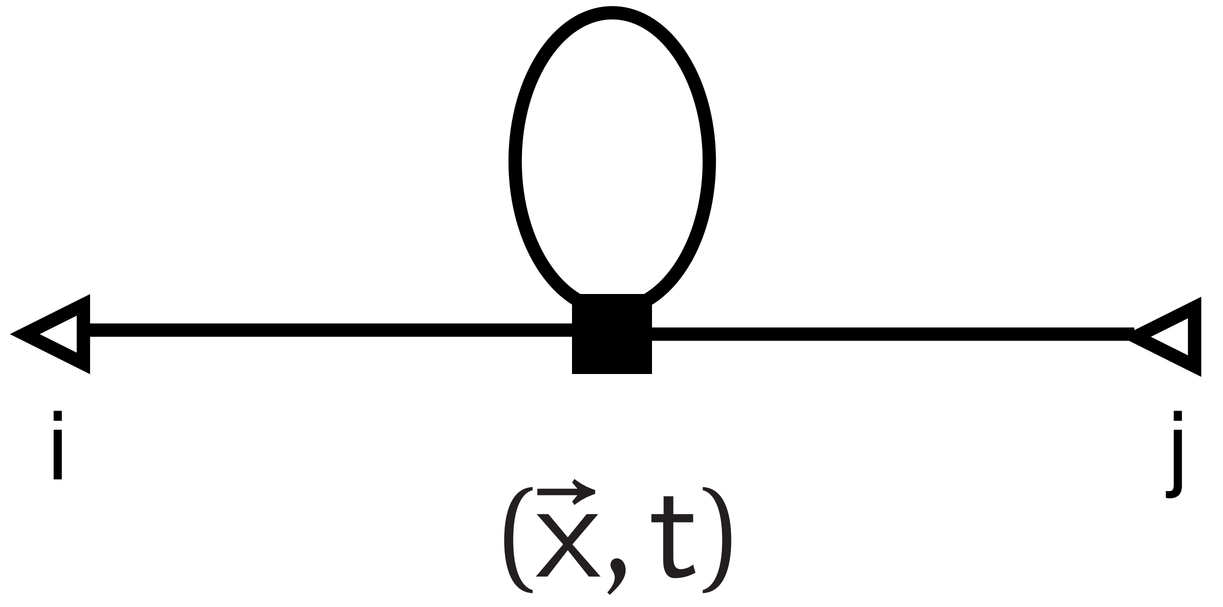
Same for other diagrams:



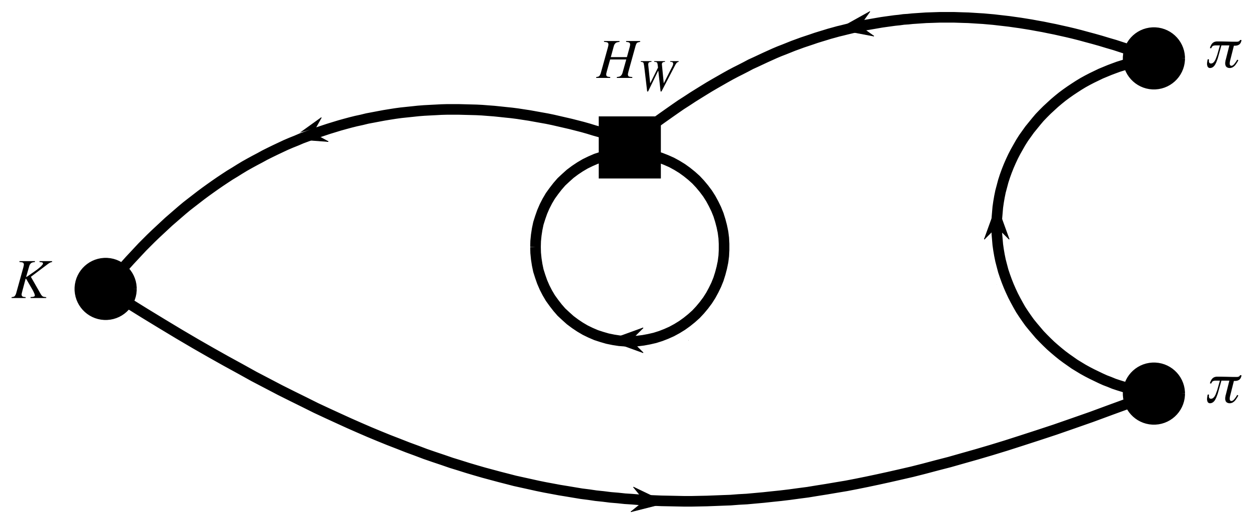
Contraction for eye diagrams

- Calculate meson field-like object

$$\Pi^{4q}_{ij}(t) = \sum_{\vec{x}}$$



Doing GPU optimization for 64^3x128 lattice calculation



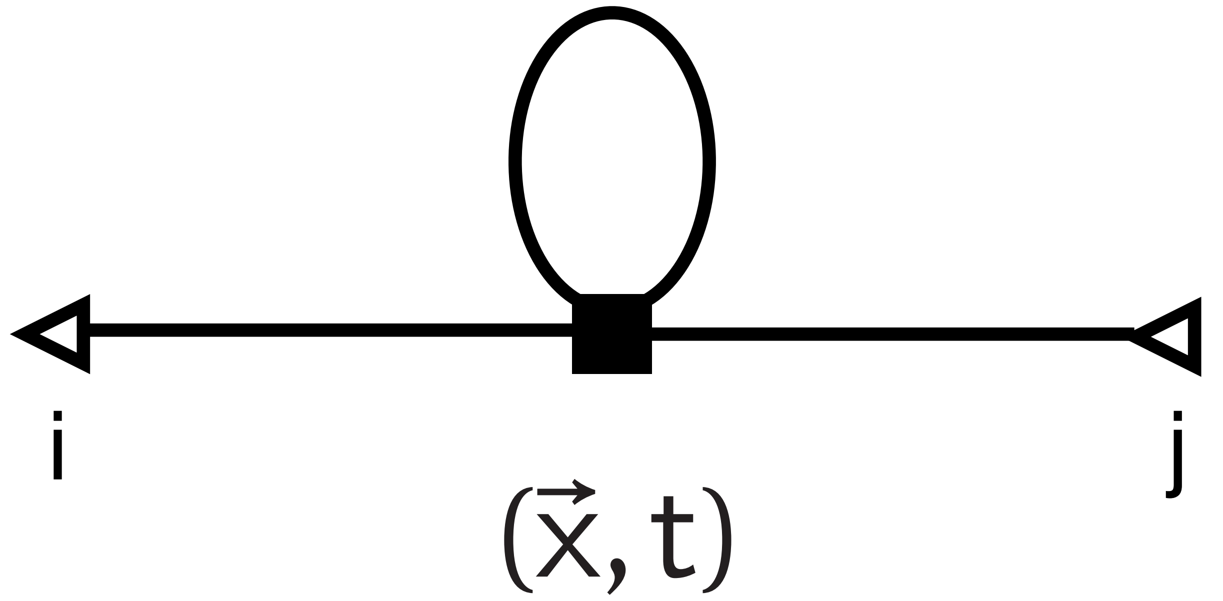
$$= \sum_{i,j,k,l} \Pi^K_{ij}(t_K)\Pi^{4q}_{jk}(t_{4q})\Pi^{\pi}_{kl}(\vec{p}, t^{\pi}_1)\Pi^{\pi}_{li}(-\vec{p}, t^{\pi}_2)$$

$$= \sum_{i,j} \Pi^K_{ij}(t_K)\Pi^{4q}_{ji}(t_{4q}) \cdot \sum_{k,l} \Pi^{\pi}_{kl}(\vec{p}, t^{\pi}_1)\Pi^{\pi}_{lk}(-\vec{p}, t^{\pi}_2)$$

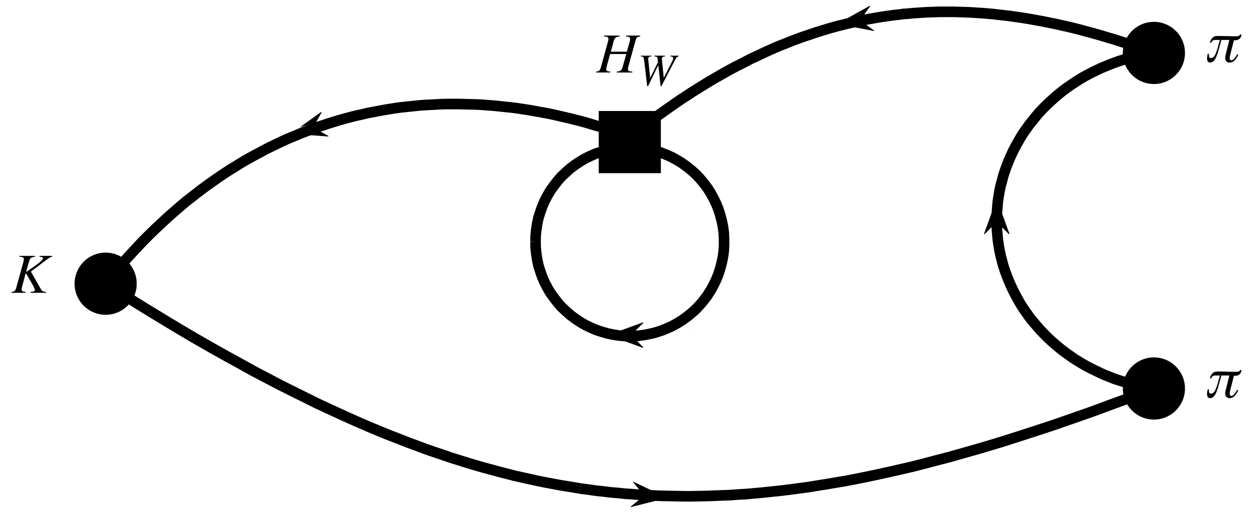
Contraction for eye diagrams

- Calculate meson field-like object

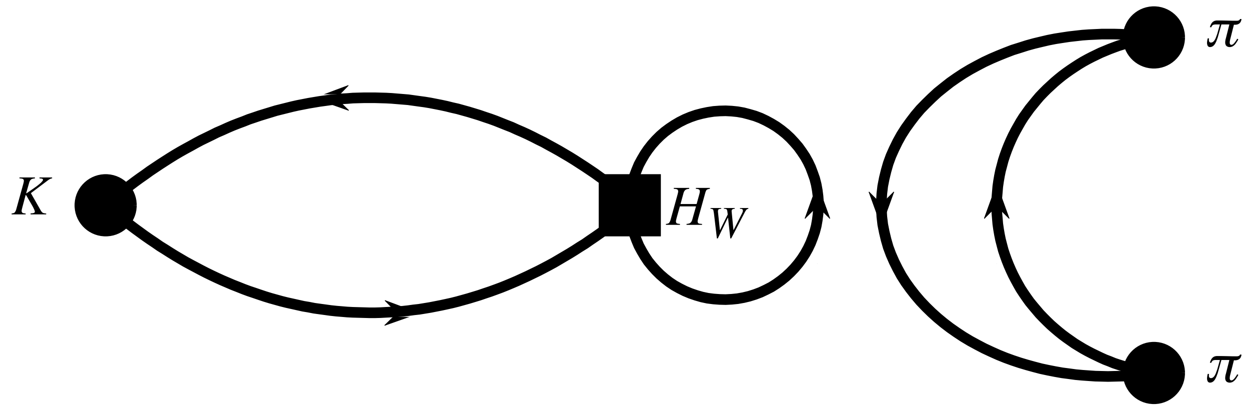
$$\Pi^{4q}_{ij}(t) = \sum_{\vec{x}}$$



Doing GPU optimization for 64^3x128 lattice calculation



$$= \sum_{i,j,k,l} \Pi^{K}_{ij}(t_K)\Pi^{4q}_{jk}(t_{4q})\Pi^{\pi}_{kl}(\vec{p}, t^{\pi}_1)\Pi^{\pi}_{li}(-\vec{p}, t^{\pi}_2)$$



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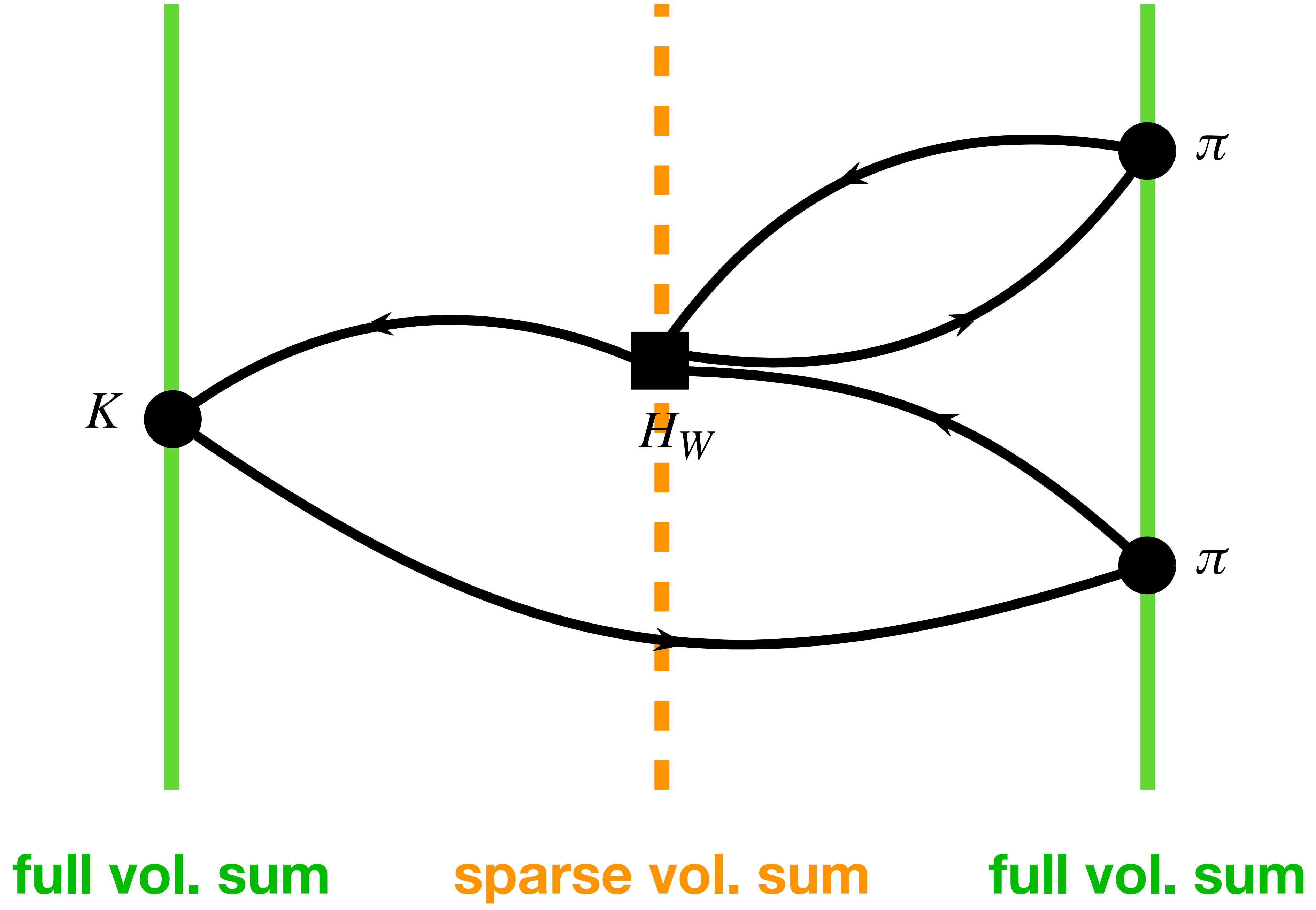
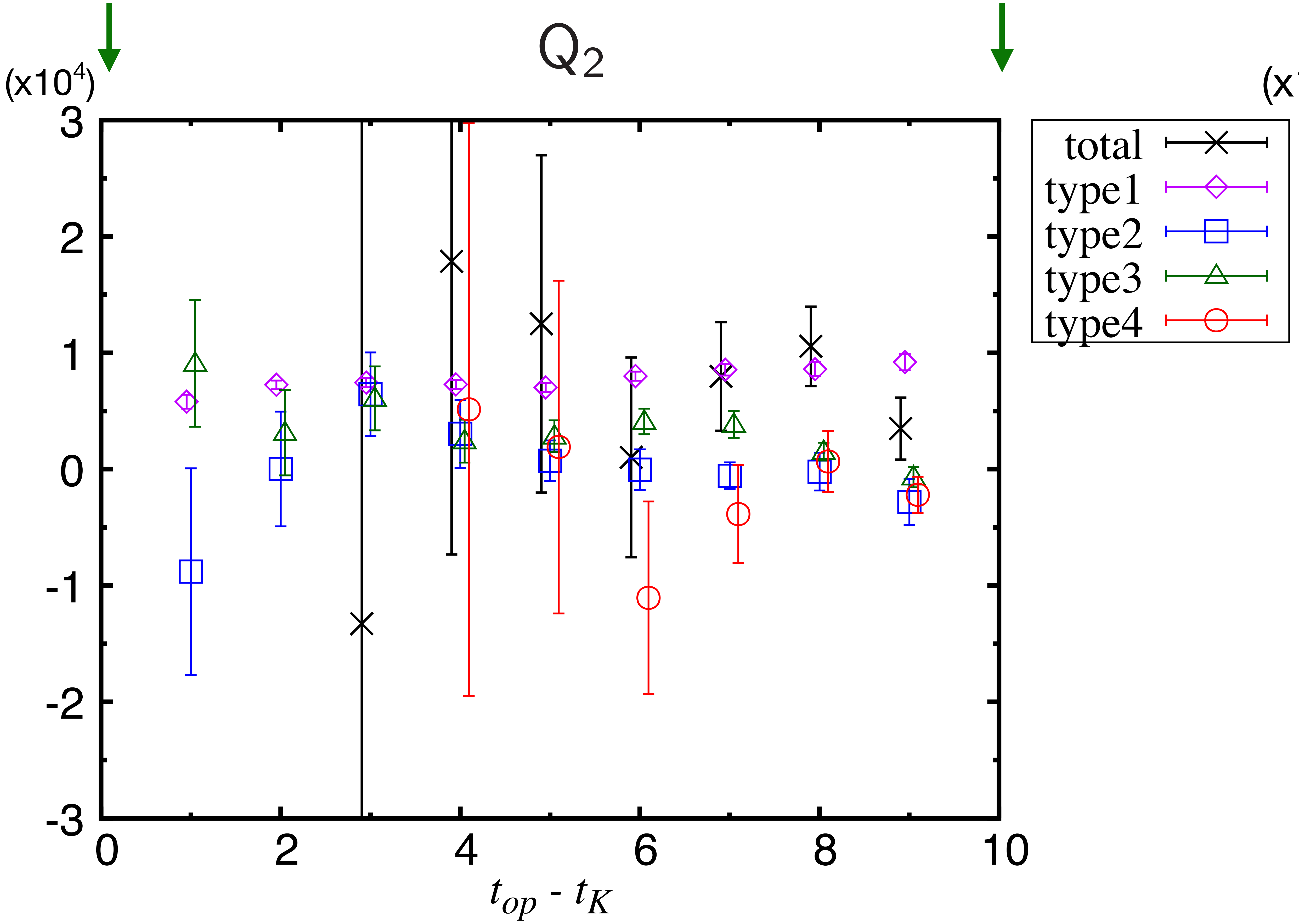
Contraction for connected diagrams

- Connected diagrams more expensive
  - Cost proportional to volume of Q_i
  - but statistically much more precise
- reduce average volume by factor of 64 for spatial volume and 8 for time translation

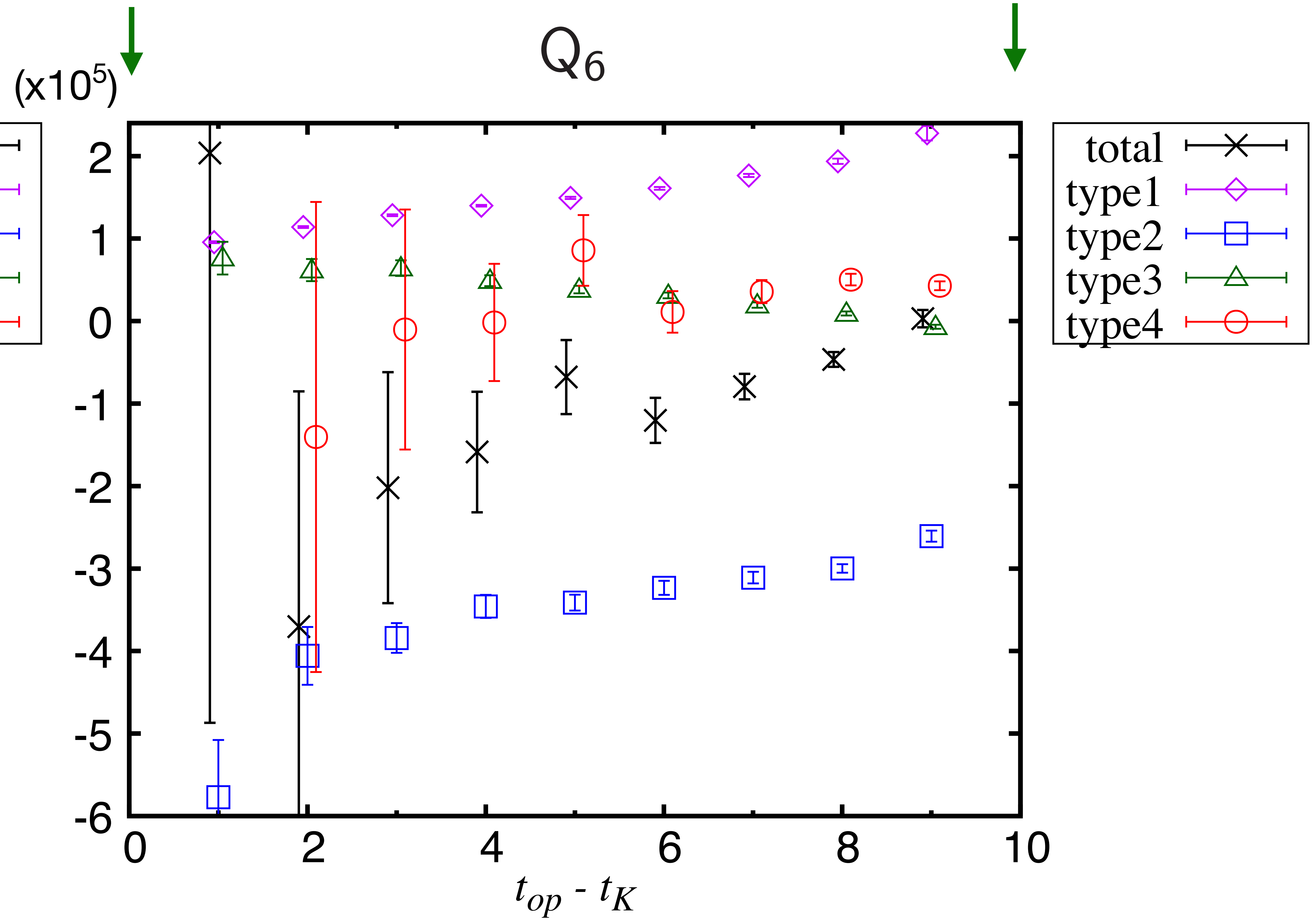


I = 0 correlation functions

kaon operator ↓ **$\pi_{p=(0,0,1)}\pi_{p=(0,0,-1)}$ operator** ↓



kaon operator ↓ **$\pi_{p=(0,0,1)}\pi_{p=(0,0,-1)}$ operator** ↓



- Sparsen H_W for types1,2 (connected) – still more precise than type4 (disconnected)

Summary

- Contractions becomes increasingly complicated as increasing number of quark propagators
- Wick contraction automated for such cases
- For $K \to \pi\pi$, not fully automated to further consider cost reductions
- Fully automated version also used for nonperturbative 3/4-flavor matching & photon structure, but the efficiency hasn’t been seriously considered

Summary

- Contractions becomes increasingly complicated as increasing number of quark propagators
- Wick contraction automated for such cases
- For $K \to \pi\pi$, not fully automated to further consider cost reductions
- Fully automated version also used for nonperturbative 3/4-flavor matching & photon structure, but the efficiency hasn't been seriously considered

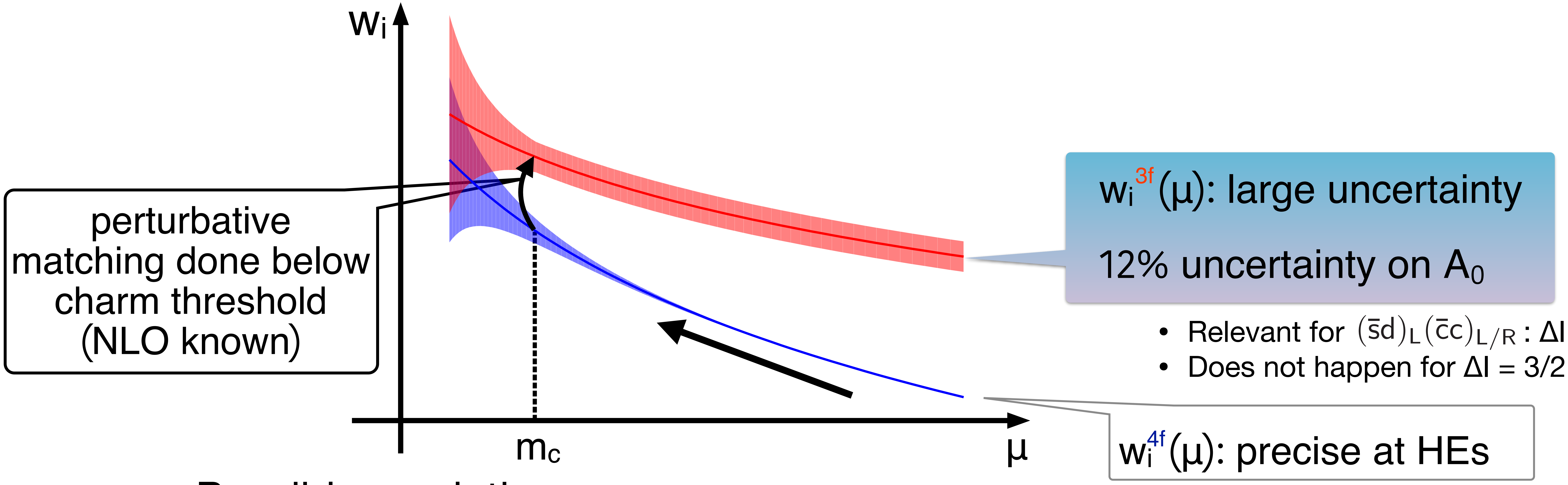
One of my dreams:

To create a fully automated measurement platform that identifies and suggests the maximally efficient approach considering various situations such as what data we already have

Nonperturbative matching of $\Delta S=1$ operators b/w 3 & 4-flavor theories

Wilson coefs

$$\langle f|H_W|i\rangle = \Sigma_i \underbrace{w_i^{3f}(\mu)}_{\text{pQCD}} \underbrace{\langle f|O_i^{3f}(\mu)|i\rangle}_{\text{LQCD}}$$

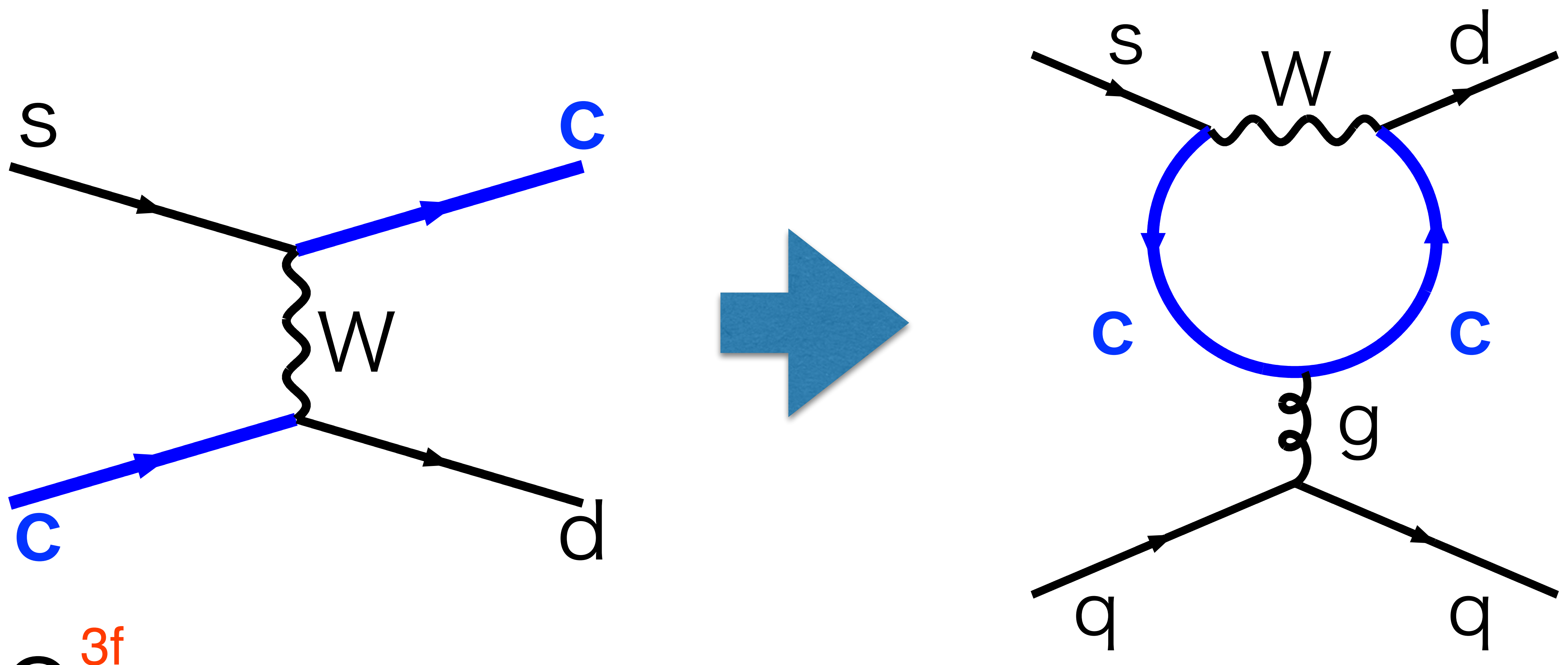


- Relevant for $(\bar{s}d)_L(\bar{c}c)_{L/R}$: $\Delta I = 1/2$
- Does not happen for $\Delta I = 3/2$

- Possible resolutions
  - NNLO perturbative matching [Cerda-Sevilla et al. Acta Phys.Polon.B 4 (2018) 1087-1096]
  - Matching nonperturbatively [MT, LATTICE2019]

How much $w_i^{3f} \neq w_i^{4f}$?

- Sea charm quark → $O(\alpha_s^2)$
- If O_i^{4f} involves charm quark…
  - WCs of charm operators will be a part of 3f WCs
  - Difference can be $O(\alpha_s)$



- w_i^{3f} necessary if MEs are calculated w O_i^{3f}
  - our fine lattice: $a^{-1} \approx 1.38$ GeV used for MEs
  - not appropriate to introduce charm on this lattice
  - $80^3 \times 160$ may be needed to have fine enough lattice maintaining m_π^{phys}

NP matching of 4-quark operators

- Basic idea

$$O_i^{4f} \to \sum_j M_{ij} O_j^{3f}$$

i.e. $\langle E_{out}|O_i^{4f}|E_{in}\rangle = \sum_j M_{ij}\langle E_{out}|O_j^{3f}|E_{in}\rangle$ for small E_{out} & E_{in} compared to m_c

- Strategy
  - Consider many 3pt functions on fine lattice

$$C_{i,ab}^{3f/4f}(t_{out}, t, t_{in}) = \langle \mathcal{O}_a(t_{out}) O_i^{3f/4f}(t) \mathcal{O}_b(t_{in})\rangle$$

  - Perform fit with many pairs of O_a & O_b at large t_{out} – t & t - t_{in} $\boxed{C_{i,ab}^{4f} = M_{ij} C_{j,ab}^{3f}}$
  - Trying with ~ 200 relevant pairs of O_a & O_b
  - Automatic Wick contractor in use

Operator basis for ΔS=1

(n_L, n_R): Representation of $SU(3)_L$ x $SU(3)_R$

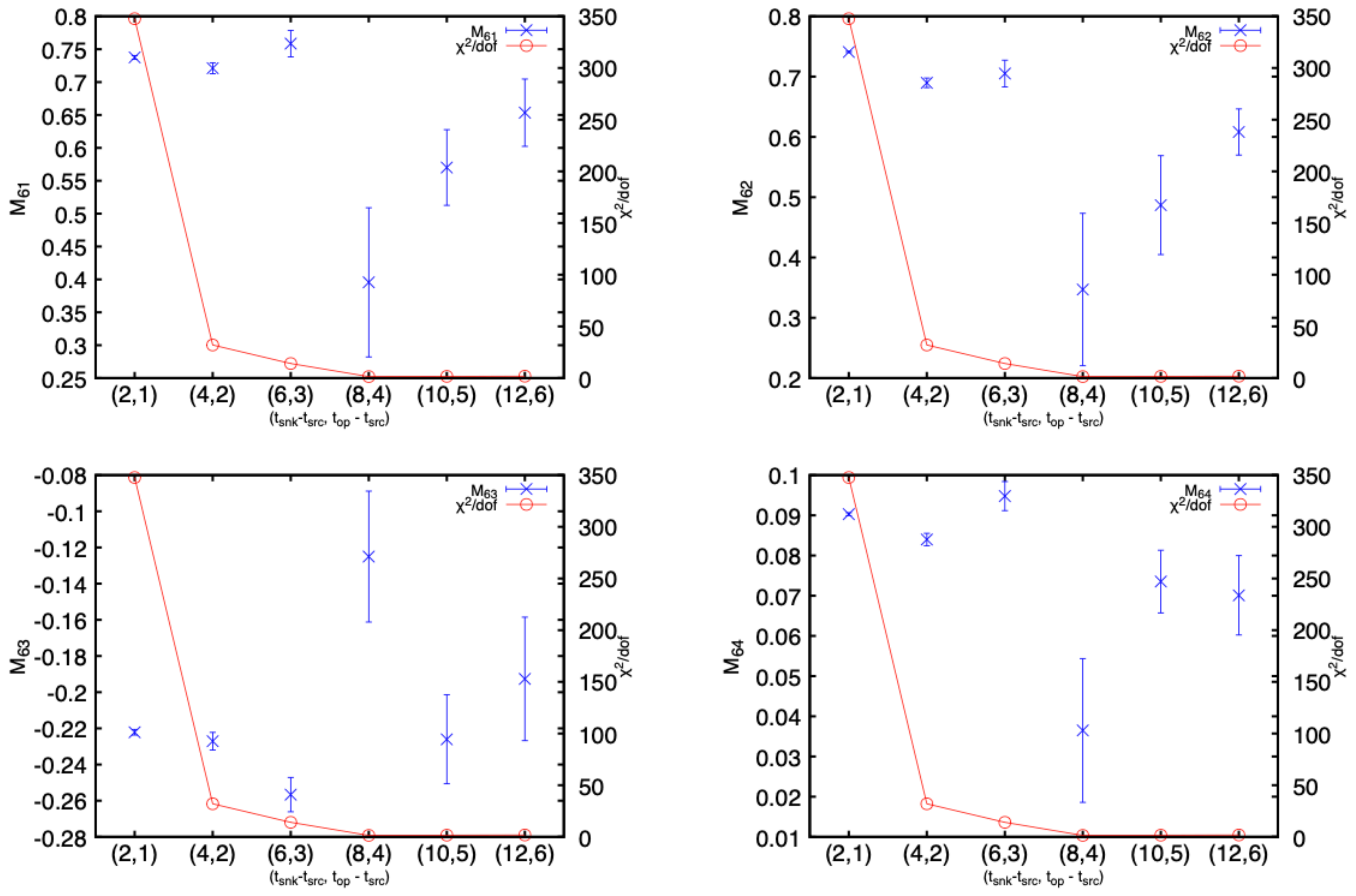
- Classification of 7 independent n_f = 3 operators
  - 1 in (27,1); 4 in (8,1); 2 in (8,8)
- 4 charm operators $(\bar{s}_\alpha d_{\alpha/\beta})_L (\bar{c}_\alpha c_{\beta/\alpha})_{L/R}$ all in (8,1) with $SU(3)_L$x$SU(3)_R$
- Only operators in (8,1) matter
  - $O_j^{3f} = (Q'_1, Q'_2, Q'_3, Q'_4)$
- $O_i^{4f} = (Q'_1, Q'_2, Q'_3, Q'_4, P_1, P_2, P_3, P_4)$

$$\Rightarrow M_{ij} = \begin{pmatrix} \mathbf{1}_{4x4} \\ \cdots\cdots\cdots \\ \bullet\ \bullet\ \bullet\ \bullet \\ \bullet\ \bullet\ \bullet\ \bullet \\ \bullet\ \bullet\ \bullet\ \bullet \\ \bullet\ \bullet\ \bullet\ \bullet \end{pmatrix}$$

16 nontrivial elements

$SU(3)_L x SU(3)_R$ to $SU(4)_L x SU(4)_R$ chiral bases

- 7-operator basis w $n_f = 3$
  - (27,1) x1, (8,1) x4, (8,8) x2
- 9-operator basis w $n_f = 4$
  - (84,1) x2, (20,1) x1, (15,1) x4, (15,15) x2
- 9x7 matching matrix can be determined once the 16 elements of M_{ij} are obtained
- Instability of M_{ij} doesn't propagate to 4-flavor matrix elements
  - Instability still concerning and trying to better understand



$SU(3)_L \times SU(3)_R$ to $SU(4)_L \times SU(4)_R$ chiral bases

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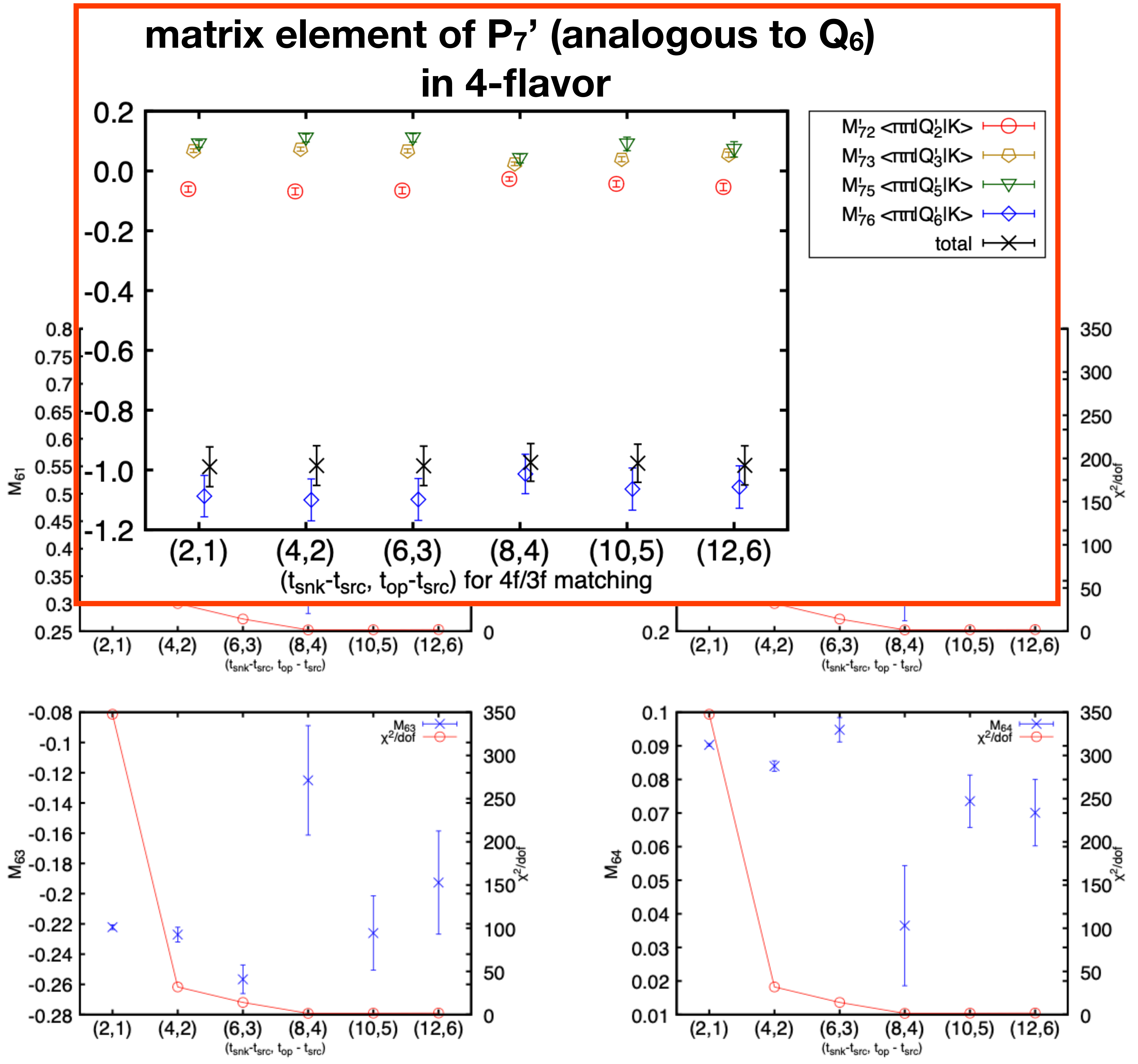


Example results on finer lattice

		value [GeV]	significant statistical errors	significant systematic errors
ReA_0	perturbative	$2.89(23)_{stat}(35)_{sys}$ x 10^{-7}	matrix elements 8.0%	Wilson coefficients 12%
	nonperturbative	$2.97(22)_{stat}(1)_{sys}$ x 10^{-7}	• matrix elements 6.5% • 3/4-flavor matching 2.3%	
ImA_0	perturbative	$-7.11(54)_{stat}(94)_{sys}$ x 10^{-11}	matrix elements 7.6%	• Wilson coefficients 12% • High-energy parameters 5.6%
	nonperturbative	$-6.82(94)_{stat}(37)_{sys}$ x 10^{-11}	• matrix elements 11% • 3/4-flavor matching 6.3%	High-energy parameters 5.4%

****outdated results**
****Discretization effects (12% overall) and finite volume effects (7% overall) not included above**

Summary

- So many issues to address for precise SM prediction of $K \to \pi\pi$
  - better continuum limit → calculation at $a^{-1} \approx 1.7$ GeV underway, 2.3 GeV planned
  - continuum limit of step scaling
  - dependence on intermediate scheme/scale on the high-energy side (μ_h)
    - NNLO perturbative matching would be very helpful
  - Nonperturbative 3/4-flavor matching mostly done, but maybe a bit more to understand before writing up a paper
  - NNLO Wilson coefficients should significantly reduce the systematic error

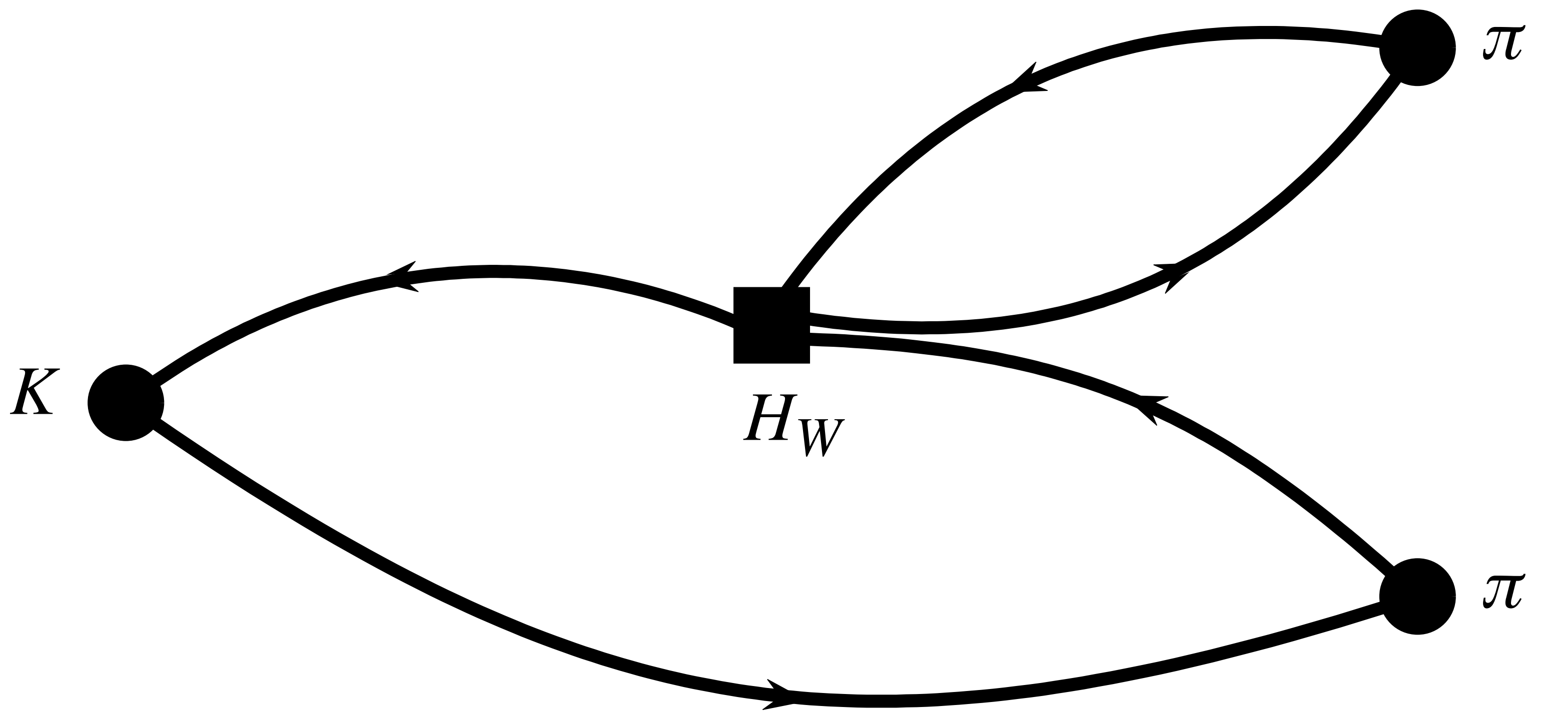
Why periodic BC?

- Already have lattice ensembles with physical pion mass
  - a^{-1} = 1 GeV, 24^3 x 64 & a^{-1} = 1.4 GeV, 32^3 x 64 & …
  - Continuum limit easier
- Hope to introduce QED/IB effects near future
  - G-parity BC violate charge conservation
  - PBC appear necessary
- Presence of $E_{\pi\pi} = 2m_\pi$ state challenging
  - S/N ratio of $E_{\pi\pi} = m_K$ state should be the same as in G-parity BC: $\sim e^{-(m_K - 2m_\pi)t}$
  - interesting to see feasibility of extracting signal of excited states

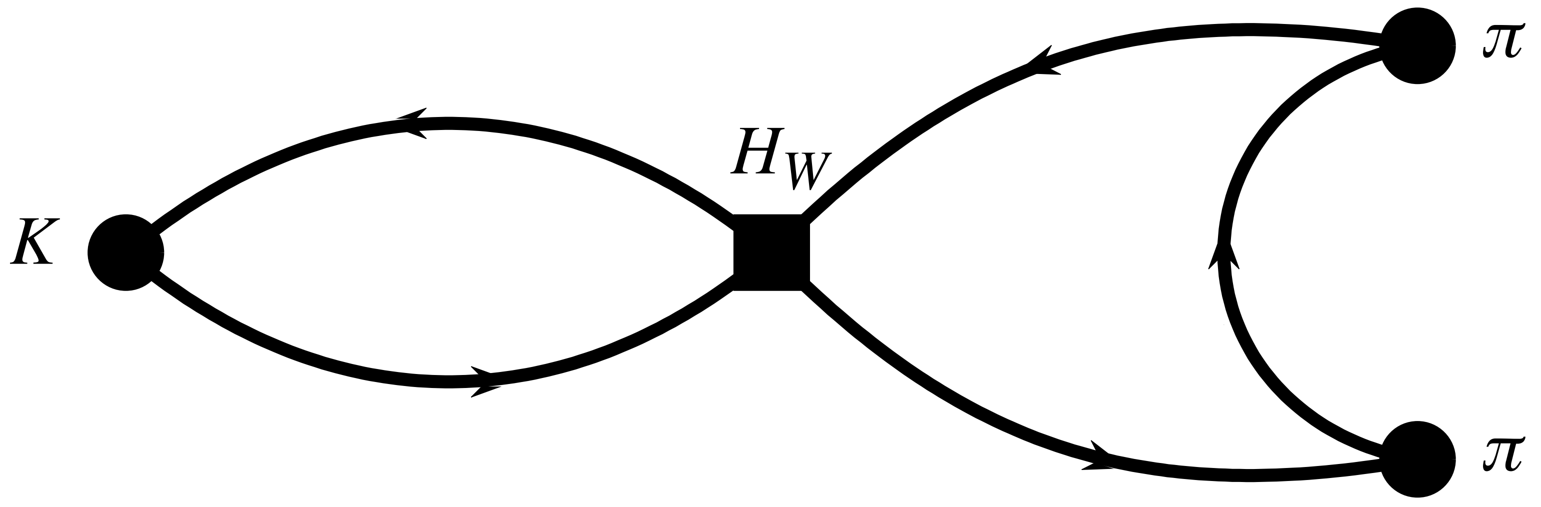
Lattice setup

- RBC/UKQCD's 2+1-flavor MDWF ensembles at physical pion & kaon masses
  - 24^3 x 64, a^{-1} = 1.0 GeV, 258 confs
  - 32^3 x 64, a^{-1} = 1.4 GeV, 107 confs
- Chiral symmetry of DWF → strong constraints on operator mixings
  - with lower-dimensional operators
  - among different representations w.r.t. chiral symmetry (8,1), (8,8) & (27,1)
- All-to-all quark propagators
  - 2,000 low modes for light quarks (no low mode for strange)
  - high-mode part: spin, color and time dilutions => 768 inversions

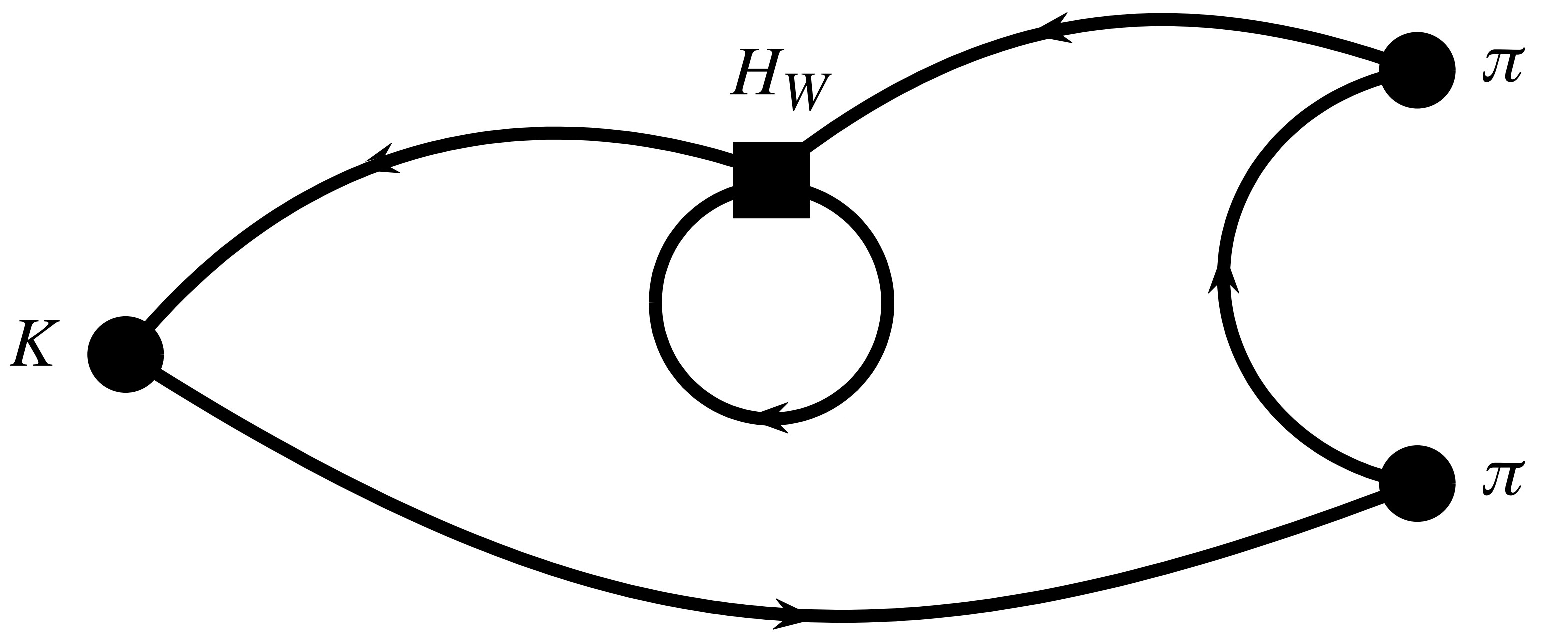
Diagrams for K → ππ 3pt functions



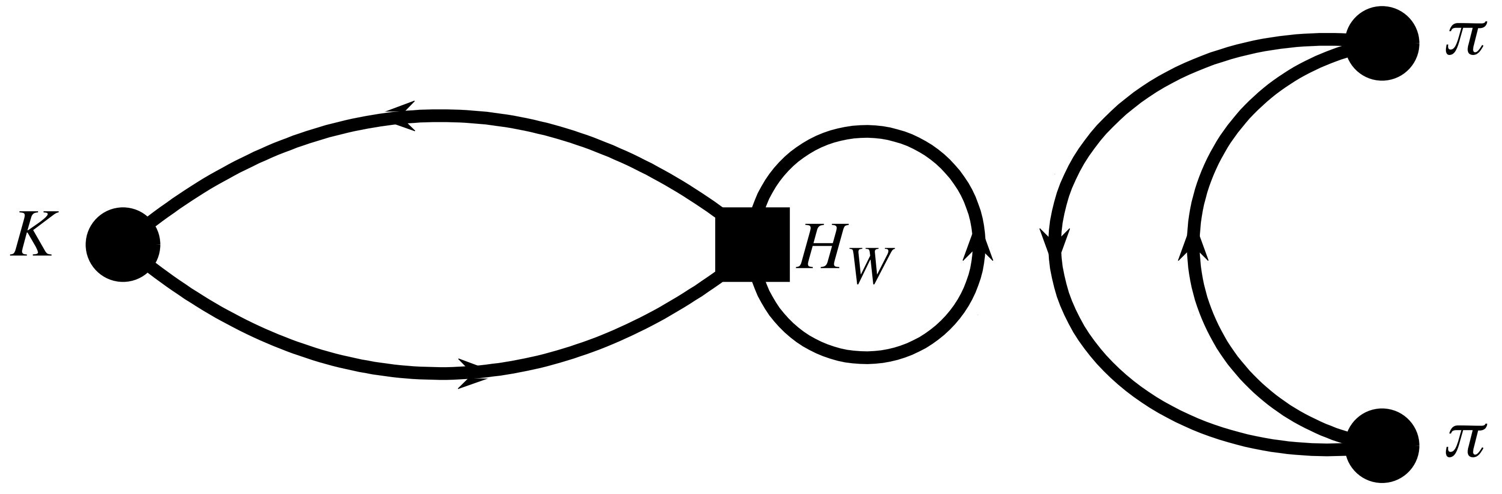
type 1



type 2



type 3



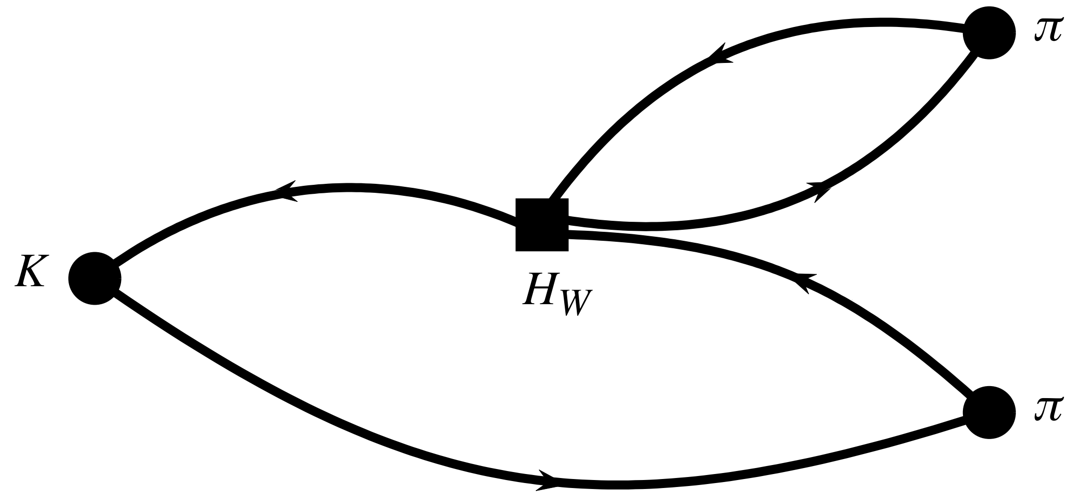
type 4

type 4 dominates stat. error

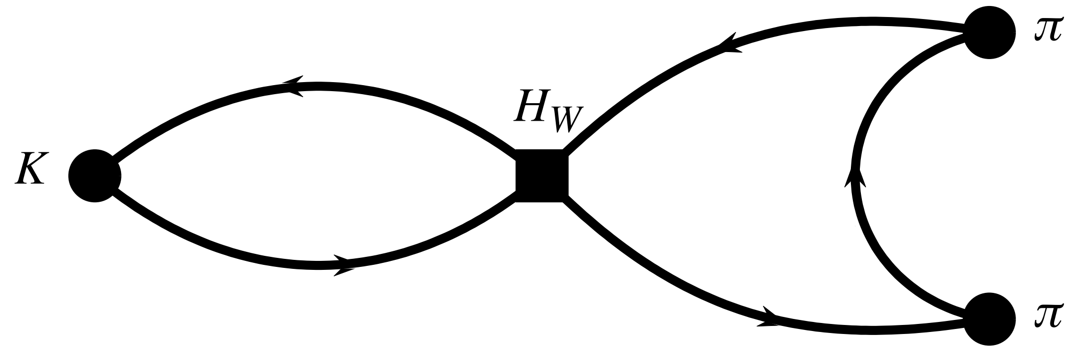
- Previous G-parity calculation
  - types 1,2: averaged over every 8 time translations
  - types 3,4: averaged over every time translation

- types 1,2 still expensive but no need of such precision

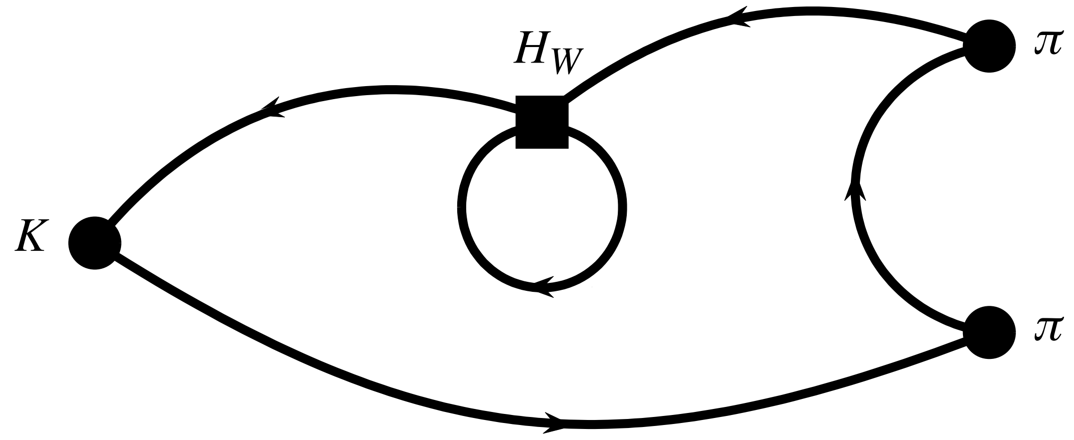
  → cost reduction?



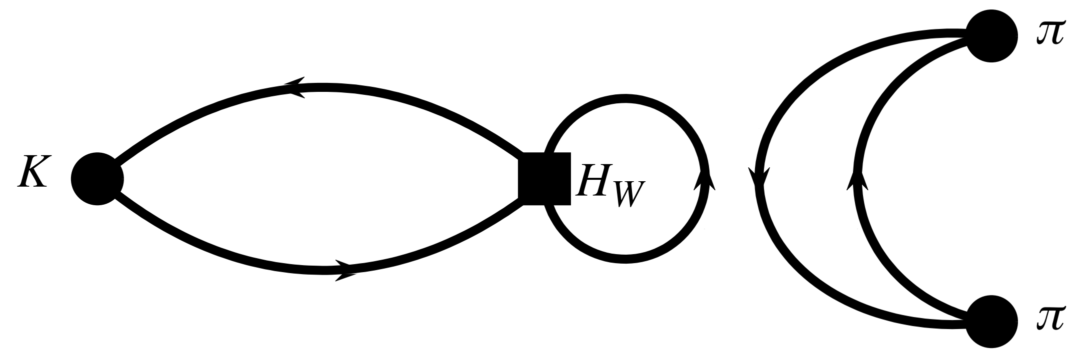
type 1



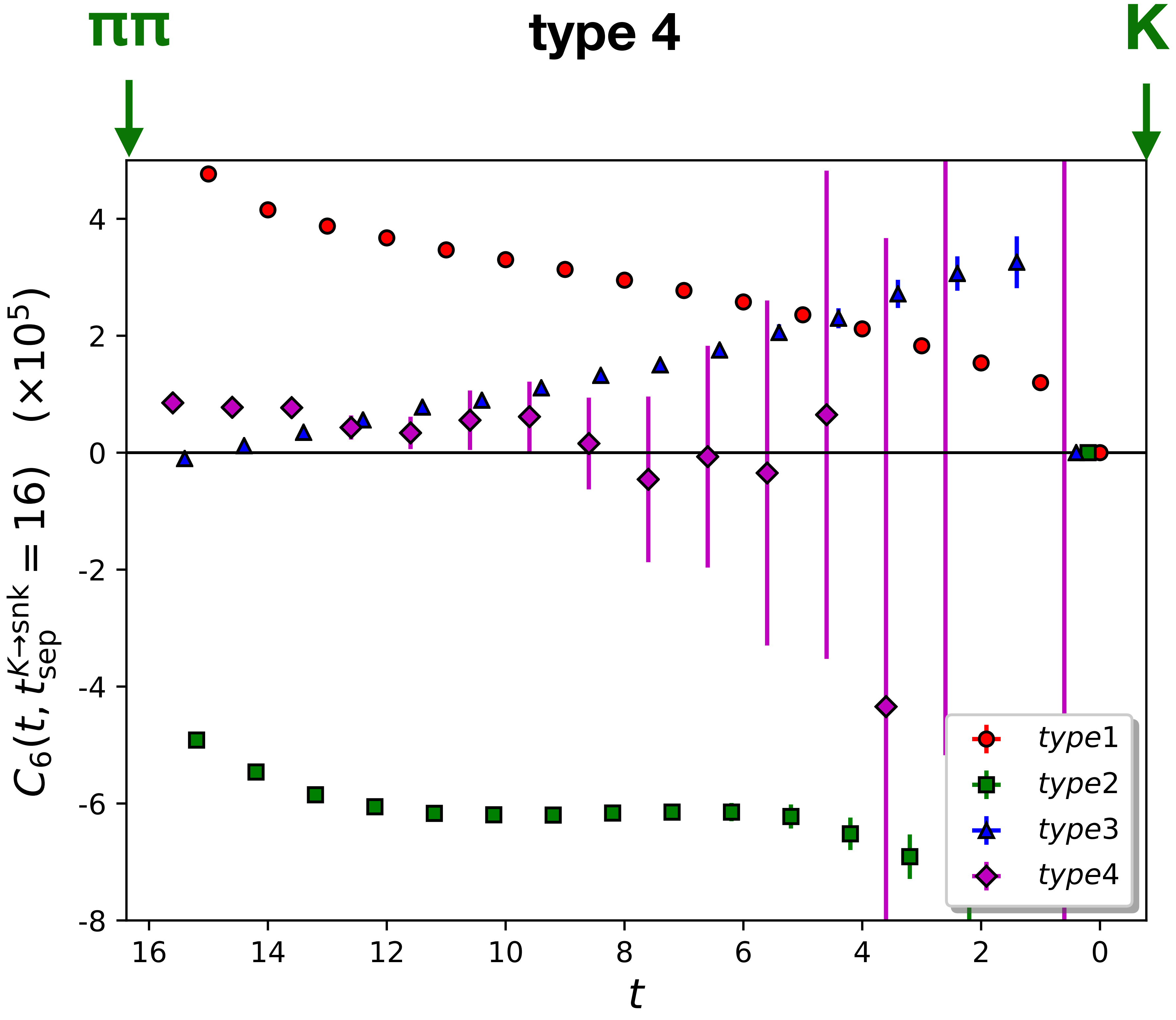
type 2



type 3

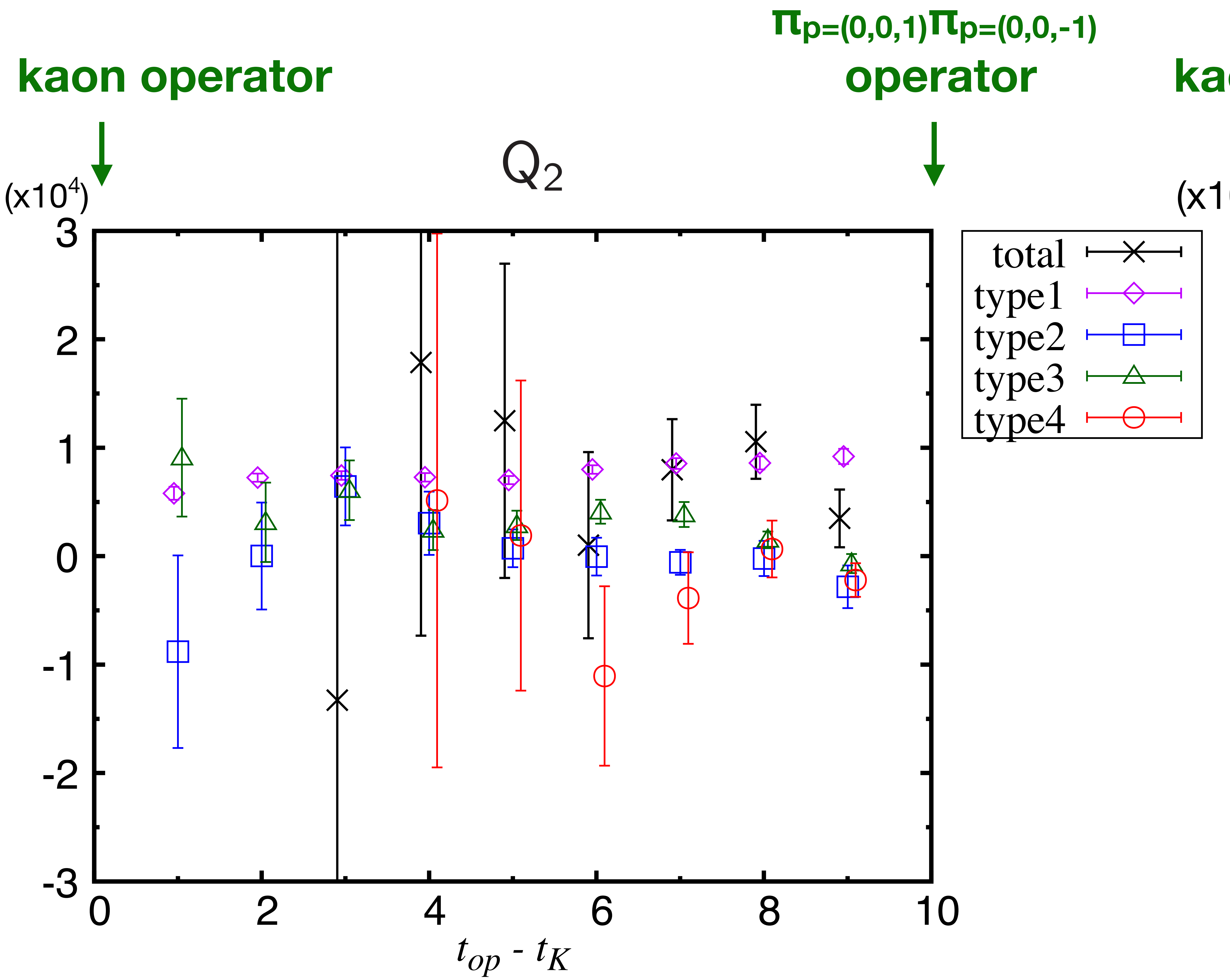


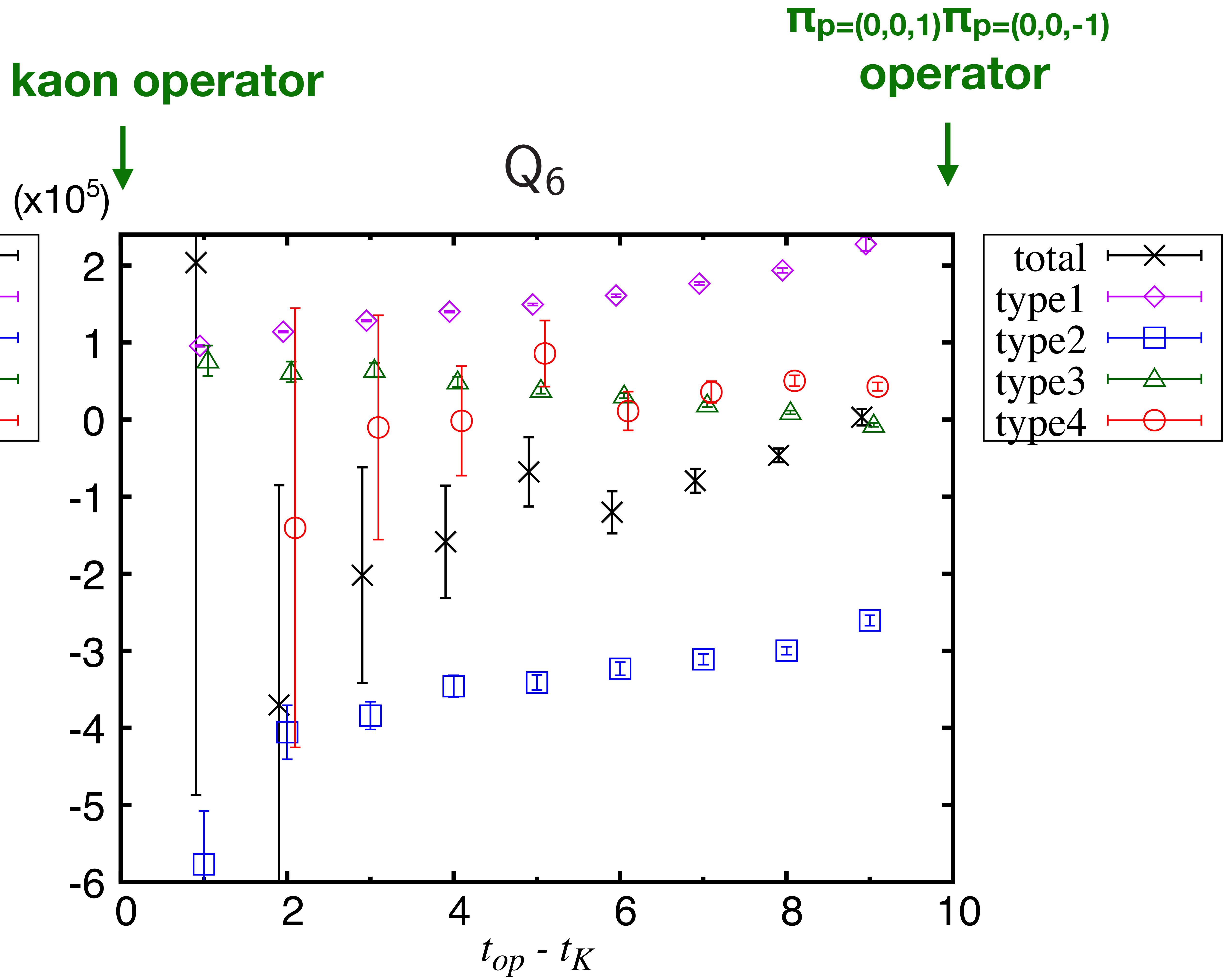
type 4



RBC/UKQCD, PRD 102,054509

I = 0 correlation functions



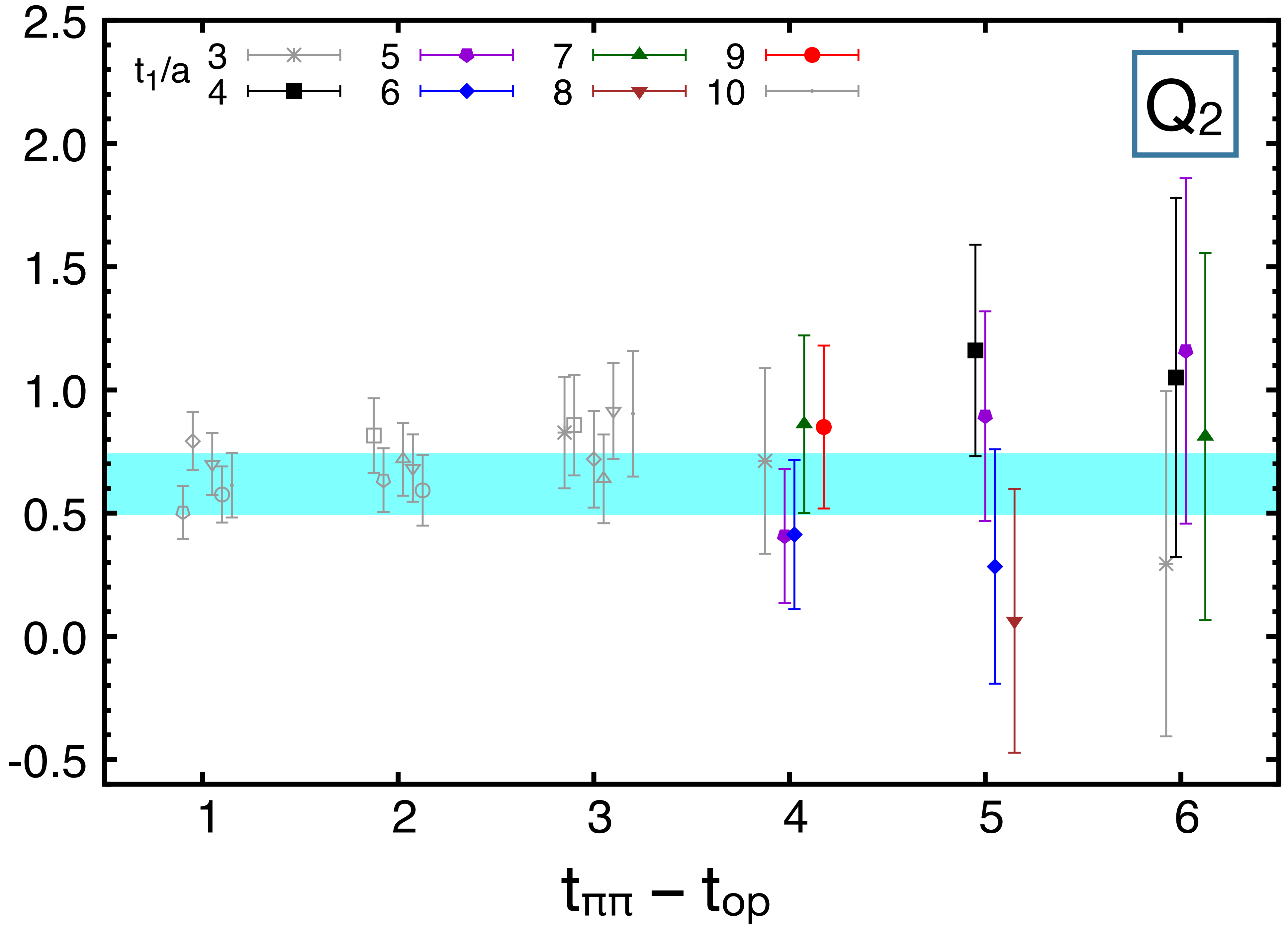


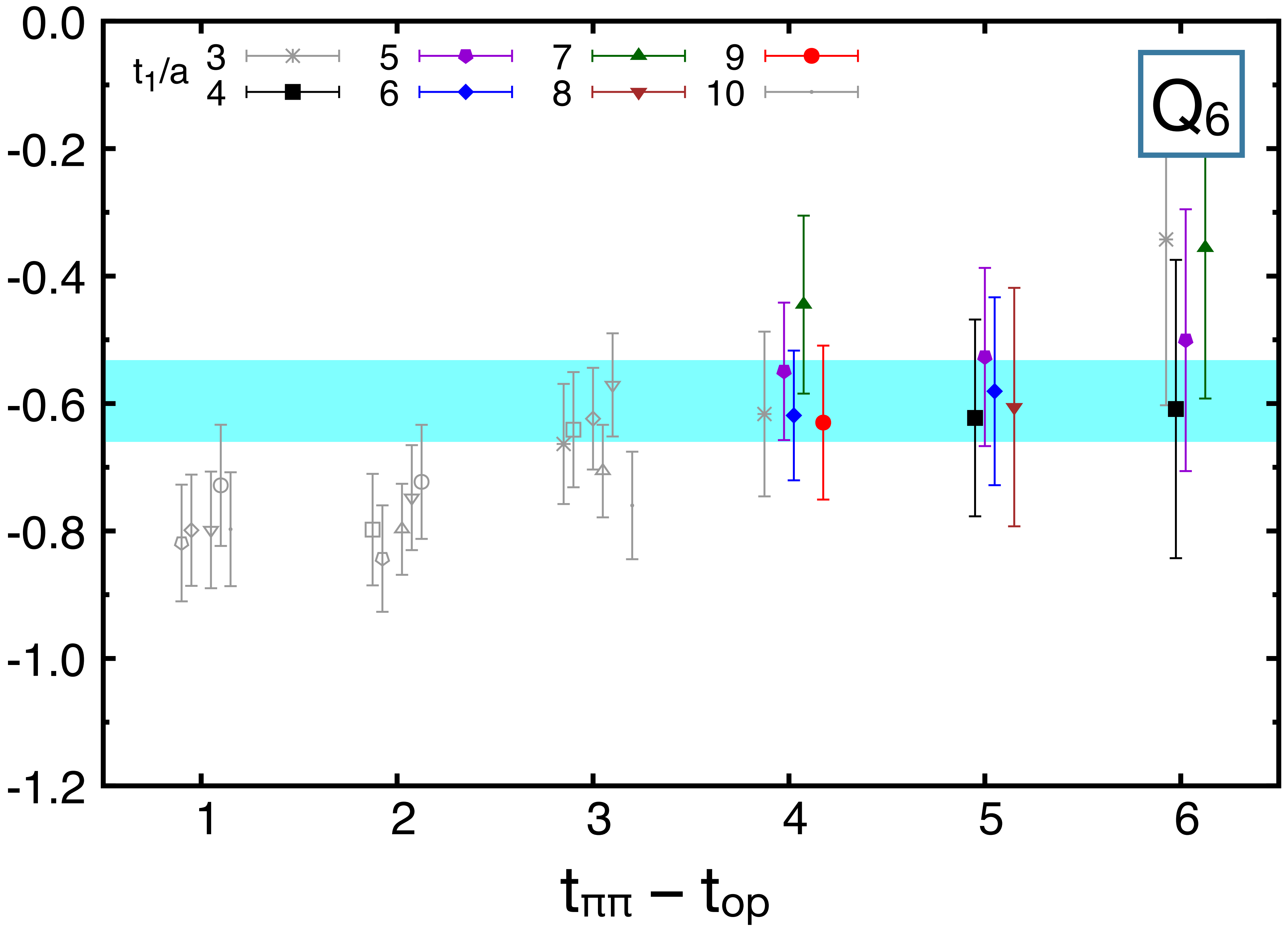
- Sparsen H_W for types1,2 – still more precise than type4
- Precision of type4 disconnected diagram

Effective matrix elements ($\Delta I = 1/2$)

- 24^3 x 64
- Plateau appears
- : Correlated fit result with

$t_1 = t_{op} - t_K \geq 4$ && $t_2 = t_{\pi\pi} - t_{op} \geq 4$ (colored filled data points)

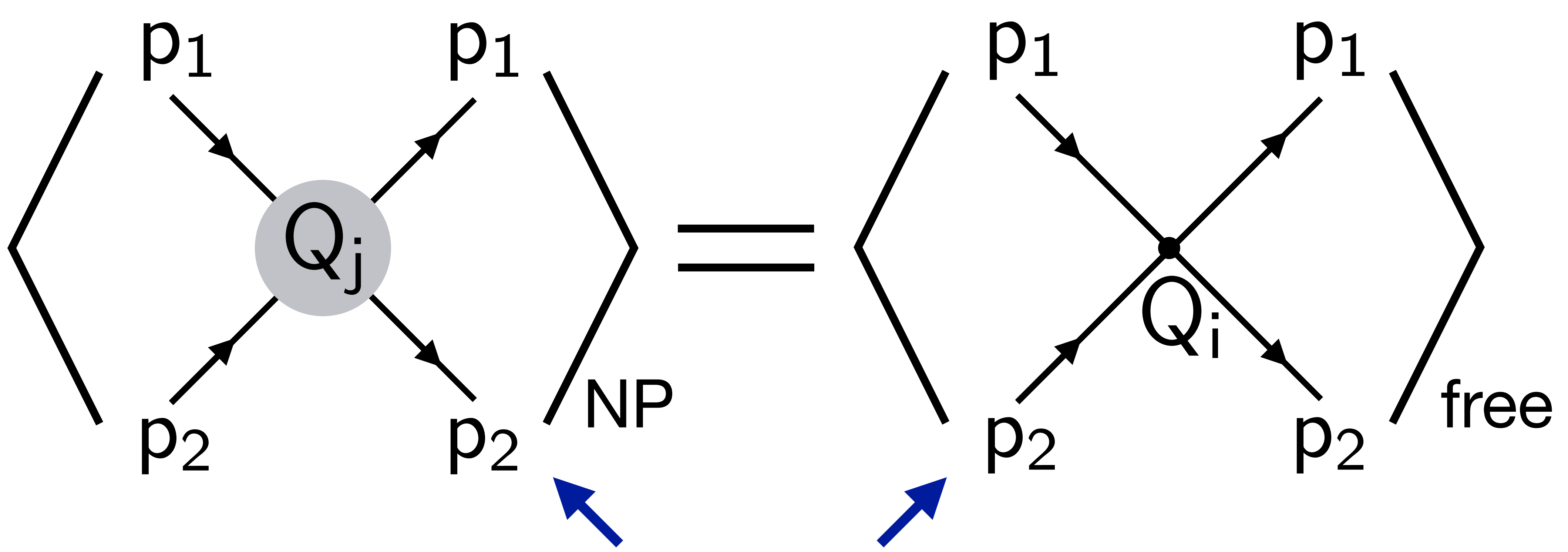




Translating to more physical ME

- Renormalization (RI/SMOM scheme)

$Z_{ij}^{RI/SMOM}(\mu')$

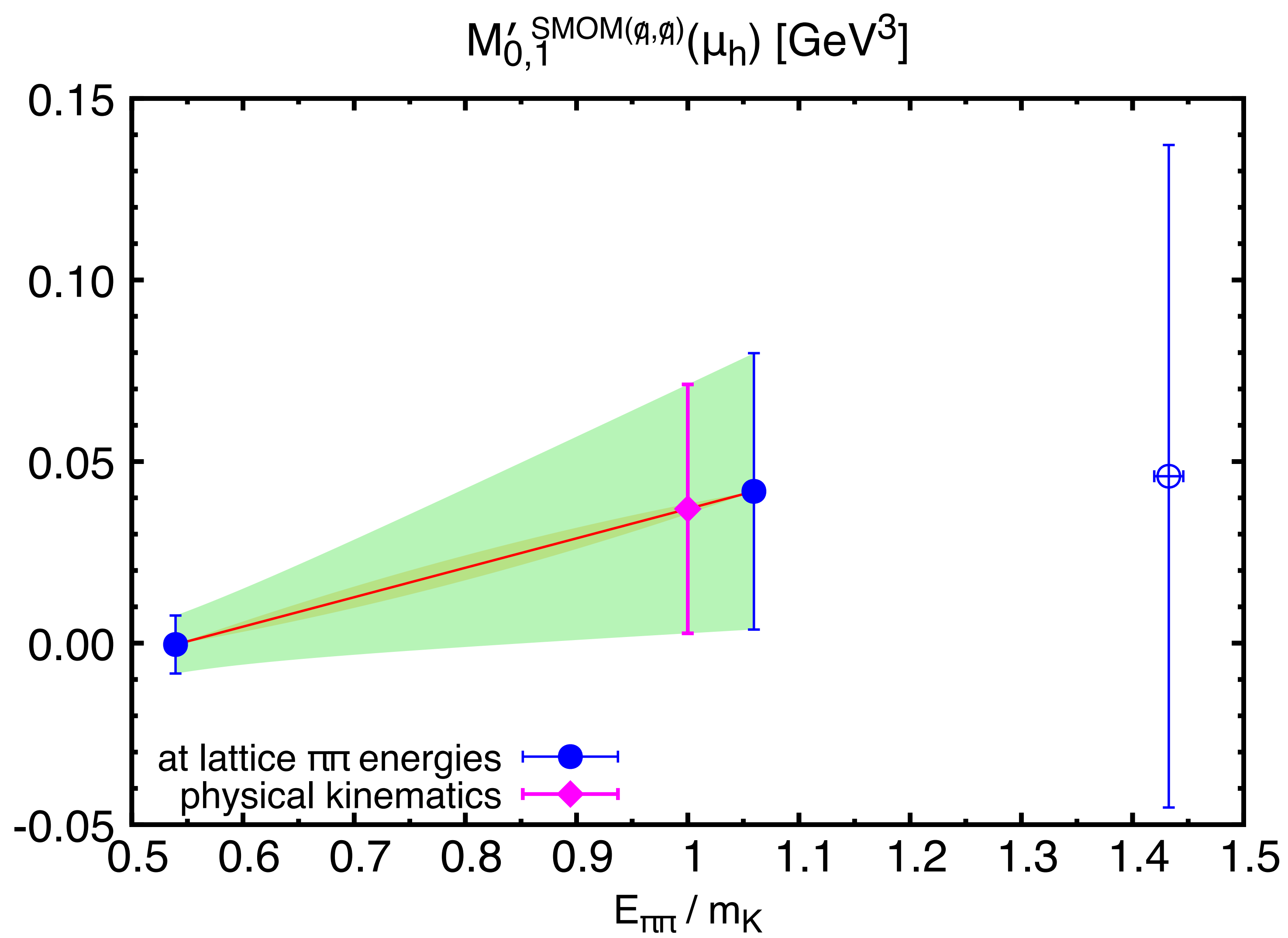


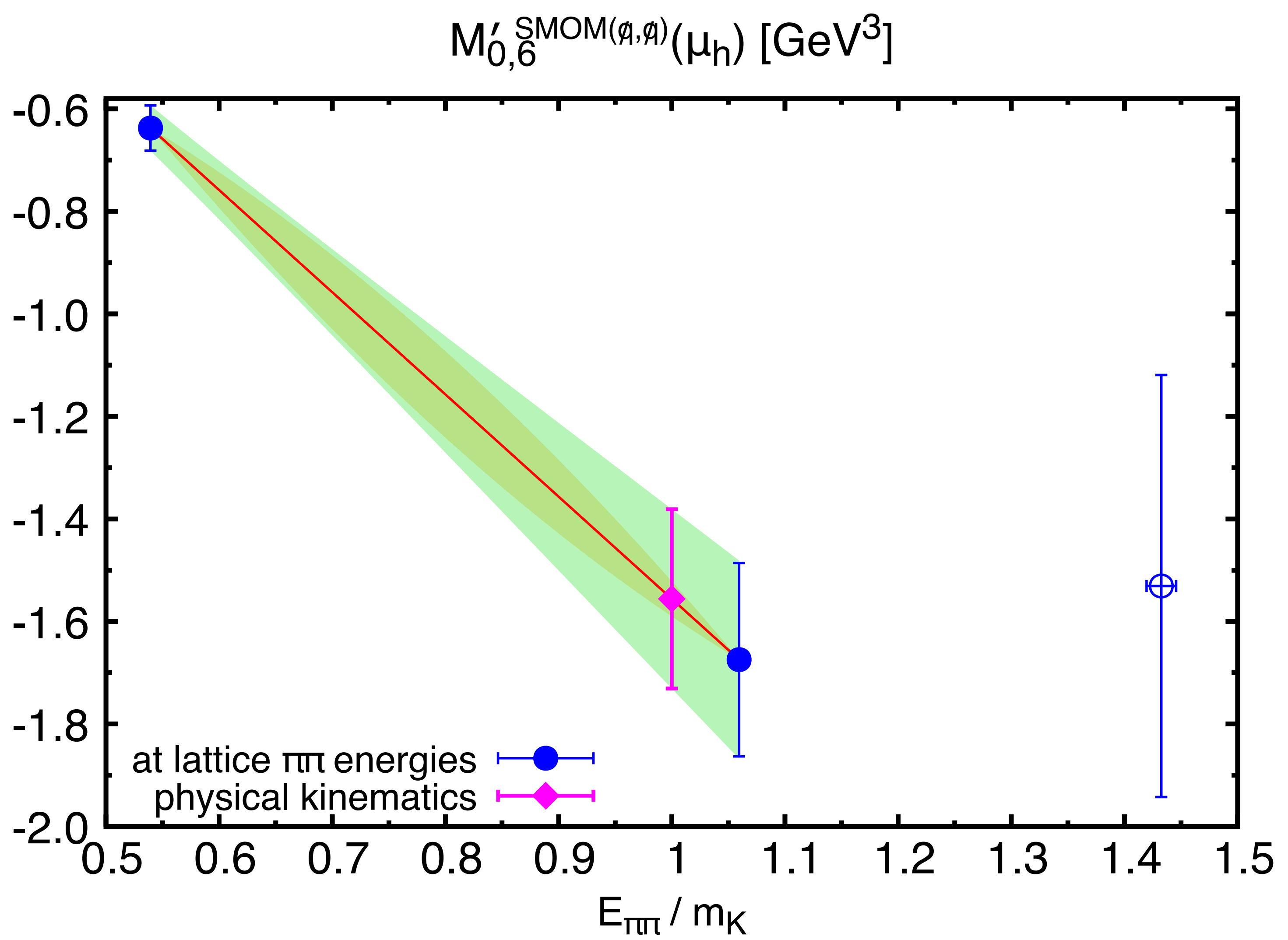
Amputated vertex functions

$$\mu'^2 = p_1^2 = p_2^2 = (p_1 - p_2)^2$$

- Interpolation

Examples of interpolation of renormalized ME
- Linear & quadratic in $E_{\pi\pi}/m_K$
- Systematic error estimated as lin vs quad is small as 1st excited st. close to on-shell





Precision performance

	32^3 G-parity BC (previous work)	24^3 Periodic BC	32^3 Periodic BC
# of configurations	741	258 → 440	107 → 470
$\Delta I = 1/2$ ME via Q_2^{lat}	10%	14% → (11%)	14% → (6.7%)
$\Delta I = 1/2$ ME via Q_6^{lat}	6.5%	8.9% → (6.8%)	11% → (5.3%)
Re A_0	11%	13% → (10%)	14% → (6.7%)

Error % (statistical) (rows $\Delta I = 1/2$ ME via Q_2^{lat}, $\Delta I = 1/2$ ME via Q_6^{lat}, Re A_0)

expectations of upcoming analysis in ( )

- Good precision performance of PBC (ME with excited-state ππ) compared to G-parity BC calculation (ME with ground-state ππ)

Summary & Outlook

- Main sources of systematic errors at the moment
  - Finite lattice spacing - *Easier to take continuum limit with PBC as we already have lattice ensembles*
  - Wilson coefficients - *Independent study on-going*
  - QED/IB effects - *Theoretical approach being developed [Christ et al, PRD106, 014508 (2021)] with PBC*
- We are successful in
  - Extracting excited-state signals of the challenging $\Delta I = 1/2$ process
  - Good precision performance of PBC approach
- Precision will reach that of experiment in the near future
  - Could attract a lot of attention