Decomposition of tensor products using noise vectors

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1 Basics of all-mode method in TRG

$$
\begin{aligned}
M_{ab} &= \sum_{s=1}^{R} u_{as}\Lambda_s v_{sb} \\
&= \sum_{s=1}^{D_{\rm svd}} u_{as}\Lambda_s v_{sb} + \sum_{s=D_{\rm svd}+1}^{R} u_{as}\Lambda_s v_{sb}
\end{aligned} \tag{1}
$$

The second term to be approximated stochastically.

2 Dilution

To approximate a unit matrix to be inserted in the second term of (1), we just need to find a set of random vectors η_α that satisfy

$$
\frac{1}{N}\sum_{\alpha=1}^{N} \eta_{\alpha;s}\eta^*_{\alpha;t} = \delta_{st} + O(1/\sqrt{N}). \tag{2}
$$

Inserting this approximated unit vector to the second term of (1) yields

$$
M_{ab} \approx \sum_{s=1}^{D_{\rm svd}} u_{as}\Lambda_s v_{sb} + \frac{1}{N}\sum_{\alpha=1}^{N}\sum_{s=D_{\rm svd}+1}^{R} u_{as}\sqrt{\Lambda_s}\eta_{\alpha;s}\sum_{s=D_{\rm svd}+1}^{R}\eta^*_{\alpha;t}\sqrt{\Lambda_t}v_{tb} \tag{3}
$$

Our original work employed purely random vectors, *i.e.* $\eta_{\alpha;s}$ randomly takes values in Z_n or U(1) with no correlation between different index points (α, s). This approach inevitably leads to a noise error on all off-diagonal elements including the one multiplied with relatively large singular values $\sqrt{\Lambda_{s=D_{\rm svd}+1}}$ and $\sqrt{\Lambda_{t=D_{\rm svd}+2}}$. (Not 100% sure, but I believe statistical error mainly comes from the off-diagonal elements with relatively large $\sqrt{\Lambda_s\Lambda_t}$.)

There should be a way to make the second term more like an importance sampling. The following idea is not quite an importance sampling, but maybe somewhat closer.

Diluted noise allows us to avoid such a large statistical noise fluctuation. We divide the set of integers $\mathcal{R} = \{D_{\rm svd}+1, \ldots, R\}$ into a few subsets. An obvious example may be

$$\mathcal{R}_i^n = \{D_{\rm svd} + n\kappa + i \leq R \,|\, \kappa \in \{0,1,2,\ldots\}\} \quad i = 1,2,\ldots,n. \tag{4}$$

where we have n subsets labeled by i. There is no intersection between $\mathcal{R}_i^n$ and $\mathcal{R}_j^n$ for $i \neq j$. Dilution ensures the noise fluctuation happens only within the same subset $\mathcal{R}_i^n$ of the index space.

We define new set of noise vectors:

$$\eta_{\alpha;s} = n\delta_{(\alpha-1) \bmod n, (s-D_{\rm svd}-1) \bmod n} \times \text{(random)}, \tag{5}$$

where (random) denotes a randomly generated number in Z_n or U(1). At least if N and $R - D_{\rm svd}$ are both divisible by n, we can derive

$$\frac{1}{N}\sum_{\alpha=1}^{N} \eta_{\alpha;s}\eta^*_{\alpha;t} = \delta_{st} + O(\sqrt{n/N})\delta_{(s-t) \bmod n, 0}. \tag{6}$$

While we now see an $O(\sqrt{n/N})$ noise error instead of $O(\sqrt{1/N})$, it is not present in every off-diagonal element. With $n \geq 2$ we can prevent the most concerning noise error from the off-diagonal element $(D_{\rm svd}+1, D_{\rm svd}+2)$.